الجامعة الأردنيـــــة
كلية الدراسات العليا

التاريـــخ: / /

نموذج رقم (١٦)
اقرار والتزام بالمعايير الأخلاقية والأمانة العلمية وقوانين الجامعة الأردنية وأنظمتها وتعليماتها لطلبة الدكتوراه

أنا الطالب: محمد ابراهيم محمد ابو اسعيد الرقم الجامعي: (9060075)
تخصص: فيزياء الكليـــــة: العلوم

عنوان الاطروحة: perturbation theory for proton-neutron scattering : perturbing the energy

اعلن بأنني قد التزمت بقوانين الجامعة الأردنية وأنظمتها وتعليماتها وقراراتها السارية المفعول المتعلقة باعداد اطروحات الدكتوراه عندما قمت شخصيا" باعداد اطروحتي وذلك بما ينسجم مع الأمانة العلمية وكافة المعايير الأخلاقية المتعارف عليها في كتابة الأطروحات العلمية. كما أنني أعلن بأن اطروحتي هذه غير منقولة أو مستلة من أطاريح أو كتب أو أبحاث أو أي منشورات علمية تم نشرها أو تخزينها في أي وسيلة اعلامية، وتأسيسا" على ما تقدم فانني أتحمل المسؤولية بأنواعها كافة فيما لو تبين غير ذلك بما فيه حق مجلس العمداء في الجامعة الأردنية بالغاء قرار منحي الدرجة العلمية التي حصلت عليها وسحب شهادة التخرج مني بعد صدورها دون أن يكون لي أي حق في التظلم أو الاعتراض أو الطعن بأي صورة كانت في القرار الصادر عن مجلس العمداء بهذا الصدد.

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PERTURBATION THEORY FOR PROTON-NEUTRON SCATTERING: PERTURBING THE ENERGY

By

Mohammed Ibrahim Abu.Al.Sayyed

Supervisor

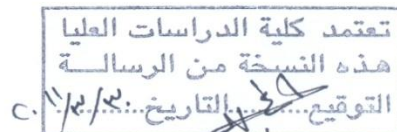
Dr. Mahmoud I. Jaghoub

This Dissertation was Submitted in Partial Fulfillment of the Requirements for the Doctor of Philosophy Degree in Physics

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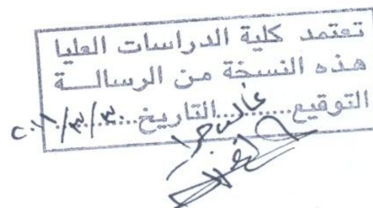
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Dedication

To my parents

To my Brothers and Sisters

Acknowledgments

I would like to thank my supervisor Dr. Mahmoud Jaghoub for his close supervision, indispensable encouragement, invaluable discussions and guidance through all stages of this work.

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Mohammed Abu.Al.Sayyed

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PERTURBATION THEORY FOR PROTON-NEUTRON SCATTERING: PERTURBING THE ENERGY

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ABSTRACT

The s-wave time-independent Schrödinger equation with an isotropic velocity-dependent potential is considered. We develop perturbation formulas for the changes in the s-wave functions and the corresponding changes in the phase shifts when the energy is changed by a small amount ΔE from an arbitrary unperturbed value E_0. The energy dependence of the phase shifts is investigated by expanding *k cot (δ)* as a power series in ΔE. At low energies, the Bethe formula for the effective range is reproduced. Three explicit examples are presented to examine the validity of the perturbation formulas. The first assumed both the local and velocity-dependent potentials to have the form of a finite square well of finite common radius b. We solved the Schrödinger equation exactly and determined the changes in the phase shifts due to small changes in the energy of the incident particle. The second assumed the local potential to be a square well, but the velocity-dependent part was taken to have a Saxon-Woods form. The third one assumed both the local and velocity dependent parts of the potential to have the form of the Saxon-Woods potential. In the second and third examples an exact analytical solution to Schrödinger equation was not possible. Consequently, the exact perturbed phase shifts were obtained numerically. The validity of the perturbation formulas was tested by comparing the exact perturbed phase shifts with the corresponding ones obtained using the derived formalism. The agreement was good for small perturbing energies as expected.

Chapter One

Introduction

1.1Introduction

A lot of what we know about nuclei and elementary particles has been discovered in scattering experiments, from Rutherford's surprise of finding that atoms have their mass and positive charge concentrated in nuclei, to the more recent discoveries, on a far smaller length scale, that protons and neutrons are themselves made up of apparently point-like quarks. The basic idea of scattering is illustrated in Figure (1-1). A particle starting from far away approaching a finite range potential, after interacting with the potential, the particle can be deflected from its original trajectory.

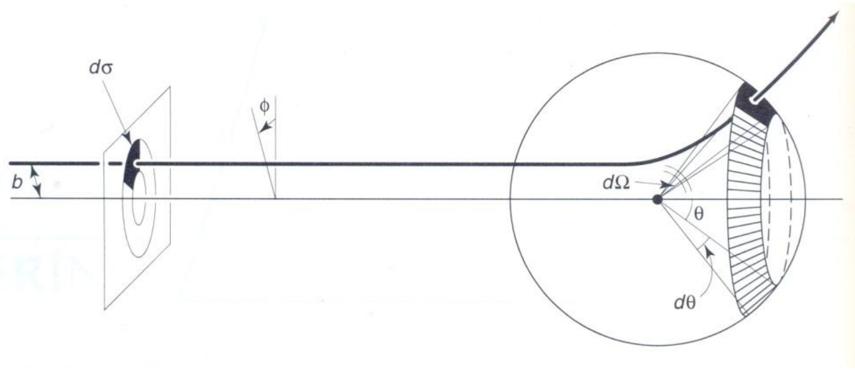


Figure (1-1): The scattering of particles by a scattering center.

In classical mechanics, collisions of two particles are entirely determined by their velocities and impact parameter (the perpendicular distance between the initial path of the projectile and the center of the field $U(r)$ created by the target). In quantum mechanics, the very wording of the problem must be changed, since in motion with definite velocities the concept of the path is meaningless, and therefore so is the impact parameter. The purpose of the scattering theory is here only to calculate the probability that as a result of the collision, the particles will deviate through any given angle. This is so-called "differential cross section" (which is the ratio of the number of particles scattered into the direction (θ, Φ) per unit time, per unit solid angle, divided by the incident flux which gives the probability that a particle is scattered through a given angle) (Bransden and Joachain,1990).

In studying the scattering theories there are two approaches, the first one is the experimental approach where one may carry out an experiment of projectiles incident on target particles and then obtain data and then analyze these data. The second one is a theoretical treatment where one may set up a mathematical model to simulate a given scattering event. In doing so, we solve a differential equation such as Schrödinger equation (Griffiths, 1995). For a given Hamiltonian the resulting differential equation may be solved exactly while other forms of the Hamiltonian result in differential equations that can be solved numerically. Sometimes we solve the problem using approximation methods, and one of these methods is the perturbation theory (Sakurai, 1994).

1.2 Previous works

Many previous works used perturbation theory to obtain analytical expressions for the changes in the wave functions and the corresponding changes in the scattering phase shifts, for example: Milward and Wilkin, 2001, derived analytical expressions for the changes in the scattering wave functions and the corresponding changes in the phase shifts by introducing a small change in a local potential. However, they performed their calculations starting from the probability density equation rather than the Schrödinger equation. The changes in the wave functions and phase shifts were shown to depend on the unperturbed wave function and the perturbing potential. In addition, M. Jaghoub, 2006, derived analytical expressions for the changes in the scattering wave functions and the corresponding changes in the scattering phase shifts when a small perturbing velocity-dependent potential is introduced. He started from Schrödinger equation and showed that knowledge of the unperturbed spectrum was not necessary and all the analytical formulas for the wave function and phase shift changes only require the knowledge of the unperturbed wave function in addition to the perturbing potential. Further, Milward and Wilkin, 2004, started from the probability density equation and studied the changes in the scattering wave functions and the corresponding changes in the

phase shifts when the incident energy is perturbed by a small amount. They conducted all their calculations for the case of a local potential. An interesting work by Romo and Valluri, 1998, studied the momentum dependence of the phase shift for local finite range and Coulomb potentials. As pointed out by the authors such studies are important as the time delay in the emergence of scattered particles is related to a change in the phase shift with respect to the wave number (Newton, 1982).

1.3 Statement of the problem

In this thesis we intend to derive analytical expressions for the changes in the s-wave functions when the energy of the incident particle is perturbed by a small amount ΔE from its initial value E_0. We shall also derive expressions for the corresponding changes in the s-wave scattering phase shifts by considering the effective range theory and expanding $k\cot(\delta)$ as a power series in the energy shift ΔE.

The Schrödinger equation shall be solved exactly for different forms of the velocity-dependent potential. For realistic nuclear potentials, closed analytical expressions for the scattering wave functions and the corresponding phase shifts may not be possible. In such cases we shall solve the Schrödinger equation numerically. In order to test the accuracy of our model, the predictions of the proposed perturbation formalism shall be compared to the exactly determined scattering phase shifts.

1.4 Thesis outline

This thesis consists of five chapters. In Chapter Two we discuss the scattering theory and the effective range expansion. In Chapter Three we shall derive the perturbation formulas for the changes in the s-wave functions and the corresponding changes in the phase shifts when small perturbations in the energy of the incident particle are introduced. In Chapter Four we

shall test the validity of the derived expressions for the changes in the phase shift in the s-wave case when small perturbations in the incident energy are present. This will be done by considering three examples; the first will assume both the local and velocity-dependent potentials to have the form of a finite square well of finite common radius b. We shall solve Schrödinger's equation exactly and determine the exact changes in the phase shifts due to the small change in the energy of the incident particle. The second assumes the local potential to be a square well, but the velocity-dependent part is taken to have a Saxon-Woods form. The third one assumes both the local and the velocity dependent parts of the potential to have Saxon-Woods forms. In the last two examples no exact analytical solution to the Schrödinger equation can be obtained. Consequently, we shall determine the exact phase shift changes numerically. Then we shall use the perturbation formulas derived in Chapter Three to calculate the changes in the phase shifts. Finally in Chapter Five we test the effectiveness of the derived formalism by comparing the exact perturbed phase shift values with the corresponding ones calculated using the perturbation formalism.

Chapter Two

Theoretical Background

Theoretical Background

2.1 Potential Scattering:

In this section we shall discuss scattering of a beam of particles by a fixed center of force (central potential). This system is described by the Schrödinger equation:

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V(r)\right]\Psi\ (\vec{r},t) = i\hbar\frac{\partial}{\partial t}\Psi\ (\vec{r},t), \qquad (2-1)$$

where m is the reduced mass and V (r) is a central potential. Experimentally, the incident beam of particles is switched on for times very long compared with the time a particle would take to cross the interaction region, so the potential V(r) does not depend on the time, hence the wave function $\Psi(\vec{r})$ satisfies the time-independent Schrödinger equation:

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V(r)\right]\psi\ (\vec{r})\ = E\psi\ (\vec{r}), \qquad (2-2)$$

where E= $\frac{\hbar^2 k^2}{2m}$ is the incident energy and k is the wave number of the particle (Bransden and Joachain, 1990).

We can rewrite equation (2-2) as:

$$[\nabla^2 + k^2 - U(r)]\psi(\vec{r}) = 0, \qquad (2-3)$$

where U(r) is the reduced potential given by $U(r) = \frac{2\,m\,V(r)}{\hbar^2}$.

For short-range potentials which either vanish beyond a certain distance or decrease exponentially with r, then for large r, $U(r)$ can be neglected and (2-3) reduces to

$$[\nabla^2 + k^2]\psi(\vec{r}) = 0\ , \qquad (2-4)$$

which is Schrödinger's equation for a free particle.

In the large r region, $\psi\ (\vec{r})$ describes both the incident beam of particles and the scattered one, so that we can write

$$\psi(\vec{r}) \xrightarrow[r\to\infty]{} \psi_{inc}(\vec{r}) + \psi_{sc}(\vec{r}). \qquad (2-5)$$

The incident particles can be described as plane waves traveling in the z-direction

$$\psi_{inc}(\vec{r}) = e^{ikz}. \qquad (2-6)$$

The scattered wave function $\psi_{sc}(\vec{r})$ must represents an outgoing spherical wave which has the form

$$\psi_{sc}(\vec{r}) = f(k,\theta,\phi)\frac{e^{ikr}}{r}, \qquad (2-7)$$

where $f(k,\theta,\phi)$ is called the scattering amplitude and gives the probability of scattering in a given direction (θ,ϕ), and hence is related to the differential cross section.

If we substitute $\psi_{inc}(\vec{r})$ and $\psi_{sc}(\vec{r})$ in equation (2-5), we see that for large r the scattering wave function must satisfy the asymptotic boundary condition

$$\psi(\vec{r}) \xrightarrow[r\to\infty]{} e^{ikz} + f(k,\theta,\phi)\frac{e^{ikr}}{r}. \qquad (2-8)$$

The probability current is a measure of the number of particles crossing a unit area perpendicular to the beam direction per unit time, and is defined through the relation

$$j(r) = \frac{\hbar}{m}\mathrm{Im}(\psi^{*}\,\nabla\psi), \qquad (2-9)$$

where ∇ in spherical polar coordinates is written as

$$\nabla = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\,. \qquad (2-10)$$

Therefore, for the incident plane wave we obtain

$$j_{inc} = \frac{\hbar}{m}\mathrm{Im}(e^{-ikz}\frac{d}{dz}e^{ikz}) = \frac{\hbar k}{m} = v_{in}\,, \qquad (2-11)$$

where the momentum $p = \hbar k$.

The second and third terms in (2-10) are small when r is large, so the current j is in the radial direction for large r. Hence using (2-7) and (2-9) we find the radial current of scattered particles to be

$$j_{sc} = \frac{\hbar}{m}\text{Im}\left(f^*(k,\theta,\emptyset)\frac{e^{-ikr}}{r}\frac{\partial}{\partial r}\left[f(k,\theta,\emptyset)\frac{e^{ikr}}{r}\right]\right) = \frac{\hbar k}{mr^2}|f(k,\theta,\emptyset)|^2$$
$$= \frac{v_{out}}{r^2}|f(k,\theta,\emptyset)|^2.. \qquad (2-12)$$

Now, the differential cross section is defined by

$$\frac{d\sigma}{d\Omega} = \frac{r^2 j_{sc}}{j_{inc}}. \qquad (2-13)$$

Using the above expressions for the incident and scattered current densities and noting that for elastic scattering the velocities in the internal and external regions are the same, we obtain

$$\frac{d\sigma}{d\Omega} = |f(k,\theta,\emptyset)|^2. \qquad (2-14)$$

Evidently, the differential cross section is equal to the absolute square of the scattering amplitude. So theoreticians try to determine the scattering amplitude, since if we get $f(k,\theta,\emptyset)$ we can calculate the differential cross section, the scattering phase shift, and the total cross section. The scattering amplitude can be determined using the method of partial wave analysis.

2-2 Partial wave analysis:

In this section we study interactions governed by a central potential V(r). The system is symmetric about z-axis (we assume the direction of incidence to be along the z-axis) so the wave function and the scattering amplitude depend only on θ (no ϕ dependence). So we can expand ψ (r, θ) in series of Legendre polynomials:

$$\psi(r,\theta) = \sum_{\ell=0}^{\infty} R_\ell(k,r)P_\ell(\text{Cos}\theta), \qquad (2-15)$$

and

$$f(k,\theta) = \sum_{\ell=0}^{\infty} f_\ell(k)P_\ell(\text{Cos}\theta), \qquad (2-16)$$

where $f_\ell(k)$ is the partial wave amplitude. Each term of the series in (2-15) is called a partial wave. To find the equation satisfied by the radial function $R_\ell(k,\mathrm{r})$ we start from equation (2-2), and express the Laplacian operator ∇^2 in spherical coordinate as

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2 sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2 sin^2\theta}\frac{\partial^2}{\partial\phi^2}. \qquad (2-17)$$

But,

$$L^2 = -\hbar^2\left[\frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]. \qquad (2-18)$$

then we can write (2-2) as

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{L^2}{\hbar^2 r^2} - \frac{2m}{\hbar^2}V(r)\right]\psi(\vec{\mathrm{r}}) = -\frac{2m}{\hbar^2}\mathrm{E}\,\psi(\vec{\mathrm{r}}). \qquad (2-19)$$

We can look for solutions of Schrödinger equation having the separable form

$$\psi_{klm}(\vec{\mathrm{r}}) = R_\ell(k,r)Y_\ell^m(\theta,\phi), \qquad (2-20)$$

where $R_\ell(k,r)$ is the radial wave function and $Y_\ell^m(\theta,\phi)$ are the spherical harmonics. Inserting (2-20) into (2-19) and using $L^2Y_\ell^m(\theta,\phi) = \ell(\ell+1)\hbar^2Y_\ell^m(\theta,\phi)$ we obtain for the radial wave function the differential equation

$$\left[\frac{d^2}{dr^2} + \frac{2}{\mathrm{r}}\frac{d}{dr} - \frac{\ell(\ell+1)}{r^2} - U(r) + k^2\right]R_\ell(k,\mathrm{r}) = 0, \qquad (2-21)$$

The reduced wave function $v_l(k,r) = r\,R_l(k,r)$ satisfies the following radial equation

$$\frac{d^2v_\ell(k,\mathrm{r})}{dr^2} + \left[k^2 - U(r) - \frac{\ell(\ell+1)}{r^2}\right]v_\ell(k,\mathrm{r}) = 0. \qquad (2-22)$$

For r > b (where b is the range of the potential beyond which it identically vanishes) the equation satisfied by $R_\ell(k,r)$ becomes

$$\left[\frac{d^2}{\mathrm{d}r^2} + \frac{2}{\mathrm{r}}\frac{d}{dr} - \frac{\ell(\ell+1)}{r^2} + k^2\right]R_\ell(k,\mathrm{r}) = 0. \qquad (2-23)$$

For each ℓ the solutions of (2-23) are the spherical Bessel and Neumann functions, $j_\ell(kr)$ and $n_\ell(kr)$, respectively. The general solution is a linear combination of these

functions so that the radial function $R_\ell(k,\mathrm{r})$ in the asymptotic region in which the potential vanishes is given by

$$R_\ell(k,r) = B_\ell(k) j_\ell(kr) + C_\ell(k) n_\ell(kr), \quad r > b \qquad (2-24)$$

where $B_\ell(k)$ and $C_\ell(k)$ are independent of r.

The asymptotic forms of $j_\ell(\rho)$ and $n_\ell(\rho)$ are

$$j_\ell(\rho) \xrightarrow[\rho\to\infty]{} \frac{1}{\rho}\sin(\rho - \ell\frac{\pi}{2}), \qquad (2-25-a)$$

and

$$n_\ell(\rho) \xrightarrow[\rho\to\infty]{} -\frac{1}{\rho}\cos\left(\rho - \ell\frac{\pi}{2}\right). \qquad (2-25-b)$$

Hence, in the asymptotic region, the radial wave function is given by

$$R_\ell(k,r) \xrightarrow[r\to\infty]{} \frac{1}{kr}\left[B_\ell(k)\sin\left(kr - \ell\frac{\pi}{2}\right) - C_\ell(k)\cos\left(kr - \ell\frac{\pi}{2}\right)\right]. \qquad (2-26)$$

Equation (2-26) may be rearranged into the form

$$R_\ell(k,r) \xrightarrow[r\to\infty]{} A_\ell(k)\frac{1}{kr}\sin\left(kr - \ell\frac{\pi}{2} + \delta_\ell(k)\right), \qquad (2-27)$$

where

$$A_\ell(k) = [B_\ell^2(k) + C_\ell^2(k)]^{\frac{1}{2}}, \qquad (2-28)$$

and

$$\delta_\ell = -\tan^{-1}(C_\ell(k)/B_\ell(k)), \qquad (2-29)$$

is called the scattering phase shift which characterizes the strength of the scattering in the $\ell^{-\mathrm{th}}$ partial wave by the potential V(r).

Figure (2-1) illustrates three possible cases of the potential. The first one shows $\upsilon_\ell(k,r)$ for free particles where V(*r*) = 0 for all r. In this case $\upsilon_\ell(k,r)$ = $j_\ell(kr)$ and the phase shifts vanish for all ℓ. Note that the Neumann function $n_l(kr)$ is not included as it is not regular at the origin. The second one shows $\upsilon_\ell(k,r)$ for a finite range attractive potential (V(*r*) < 0). From the figure we see that $\upsilon_\ell(k,r)$ is pulled towards the

scattering center which in turn advances its phase making the phase shift positive. The last one shows $\upsilon_\ell(k,r)$ for repulsive potential (V(r) > 0), and we see that for a repulsive potential, $\upsilon_\ell(k,r)$ is pushed out, its phase is retarded, and the phase shift is negative.

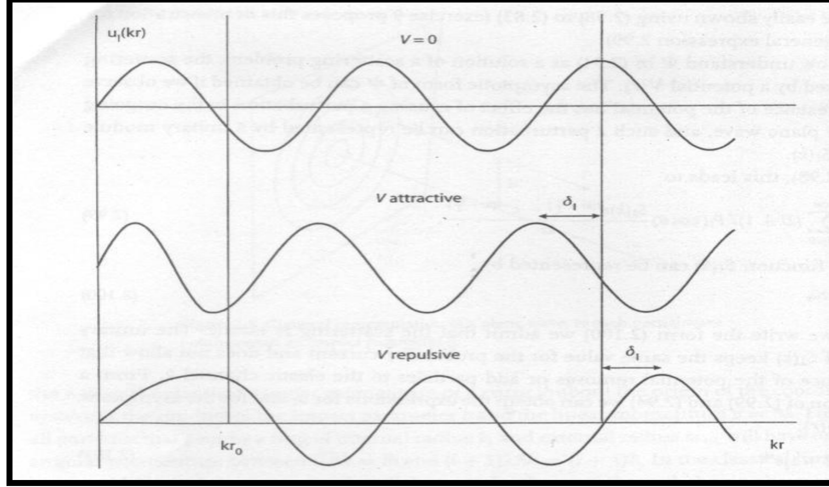


Figure (2-1) Reduced wave function for three different potentials, showing how the sign of the phase shift is determined by the function behavior in the region r < b where the potential acts.

In order to relate the phase shifts to the partial wave amplitude $f_\ell(k)$ and the scattering amplitude, we start from the asymptotic form of $\psi_\ell(\mathrm{r},\theta)$, namely;

$$\psi(\mathrm{r},\theta) \underset{\mathrm{r}\to\infty}{\longrightarrow} \mathrm{e}^{ikz} + \mathrm{f}(k,\theta)\frac{e^{ikr}}{\mathrm{r}}. \qquad (2\text{−}30)$$

But the plane wave e^{ikz} can be expanded in Legendre polynomials as

$$e^{ikz} = \sum_{\ell=0}^{\infty}(2\ell+1)\mathrm{i}^{\ell} j_\ell(kr)P_\ell(\cos\theta). \qquad (2-31)$$

Using equation (2-25-a) we obtain

$$e^{ikz} \underset{\mathrm{r}\to\infty}{\longrightarrow} \sum_{\ell=0}^{\infty}(2\ell+1)i^{\ell}(kr)^{-1}\sin\left(kr-\ell\frac{\pi}{2}\right)P_\ell(\cos\theta). \qquad (2-32)$$

Substituting (2-16) and (2-32) in (2-30) leads to

$$\psi(\mathrm{r},\theta) \underset{\mathrm{r}\to\infty}{\longrightarrow} \sum_{\ell=0}^{\infty} \frac{(2\ell+1)i^{\ell}}{kr} \sin\left(kr - \ell\frac{\pi}{2}\right) P_{\ell}(\cos\theta) + \sum_{\ell=0}^{\infty} f_{\ell}(k) p_{\ell}(\cos\theta) \frac{e^{ikr}}{r}. \qquad (2-33)$$

Or,

$$\psi(\mathrm{r},\theta) \underset{\mathrm{r}\to\infty}{\longrightarrow} \left[\sum_{\ell=0}^{\infty} \frac{(2\ell+1)i^{\ell}}{kr} \sin\left(kr - \ell\frac{\pi}{2}\right) + f_{\ell}(k)\frac{e^{ikr}}{r}\right] P_{\ell}(\cos\theta) \qquad (2-34)$$

By comparing equation (2-34) and (2-15) we see that

$$R_{\ell}(k,r) \underset{\mathrm{r}\to\infty}{\longrightarrow} \frac{(2\ell+1)i^{\ell}}{kr} \sin\left(kr - \ell\frac{\pi}{2}\right) + f_{\ell}(k)\frac{e^{ikr}}{r}. \qquad (2-35)$$

Equating (2-35) with (2-27) results in

$$A_{\ell}(k) = (2\,\ell+1) i^{\ell} e^{i\delta_{\ell}(k)}\,, \qquad (2-36)$$

and

$$f_{\ell}(k) = \frac{2\ell+1}{2ik}\left[e^{2i\delta_{\ell}(k)} - 1\right]. \qquad (2-37)$$

Consequently, the scattering amplitude $f(k,\theta)$ now becomes

$$f(k,\theta) = \sum_{\ell=0}^{\infty} f_{\ell}(k) P_{\ell}(\mathrm{Cos}\theta) = \frac{1}{2ik} \sum_{\ell=0}^{\infty} (2\ell+1)\left[e^{2i\delta_{\ell}(k)} - 1\right] P_{\ell}(\cos\theta). \qquad (2-38)$$

To illustrate the physics of scattering we rewrite equation (2-27) as

$$R_{\ell}(\mathrm{k},r) \underset{\mathrm{r}\to\infty}{\longrightarrow} \frac{-1}{2ik} A_{\ell}(k) e^{-i\delta_{\ell}(k)} \left[\frac{e^{-i\left(kr-\ell\frac{\pi}{2}\right)}}{r} - \frac{1}{r} e^{2i\delta_{\ell}(k)} e^{i\left(kr-\ell\frac{\pi}{2}\right)}\right]. \qquad (2-39)$$

The first term in (2-39) represents an incoming spherical wave while the second one is an outgoing spherical wave.

Clearly, the incoming and outgoing waves are different by the factor $e^{2i\delta_{\ell}(k)}$ which means that the effect of the potential can only produce a phase difference between the incoming and outgoing spherical waves.

The total cross section may be obtained by integrating equation (2-14) to get

$$\sigma_{tot}=\int |f(k,\theta)|^2 d\Omega = 2\pi \int_{-1}^{1} f(k,\theta)^* f(k,\theta)\, d(\cos\theta). \qquad (2—40)$$

Substituting equation (2-16) in (2-40) results in

$$\sigma_{tot} = 2\pi \sum_{\ell=0}^{\infty}\sum_{\ell'=0}^{\infty}\int_{-1}^{1} f_{\ell'}^*(k) f_\ell(k)\, P_{\ell'}(\cos\theta) P_\ell(\cos\theta)\, d(\cos\theta) \qquad (2-41)$$

Using the orthogonality property of the Legendre polynomials;

$$\int_{-1}^{1} d(\cos\theta) P_\ell(\cos\theta) P_{\ell'}(\cos\theta) = \frac{2}{2\ell+1}\delta_{\ell\ell'}, \qquad (2-42)$$

we end up with

$$\sigma_{tot} = 2\pi \sum_{\ell=0}^{\infty} f_\ell^*(k) f_\ell(k) \frac{2}{2\ell+1}\,. \qquad (2-43)$$

Equivalently, we may write:

$$\sigma_{tot} = \sum_{\ell=0}^{\infty} \frac{4\pi}{2\ell+1} |f_\ell(k)|^2. \qquad (2-44)$$

From (2-37),

$$|f_\ell(k)|^2 = \frac{(2\ell+1)^2}{k^2}\sin^2\delta_\ell. \qquad (2-45)$$

Inserting (2-45) into (2-44) leads to

$$\sigma_{tot} = \frac{4\pi}{k^2}\sum_{\ell=0}^{\infty}(2\ell+1)\sin^2\delta_\ell\ . \qquad (2-46)$$

2-3 Calculation of the phase Shift:

The computation of the phase shift can be illustrated by means of the simple square well potential. The simplified attractive nuclear potential between a proton and a neutron takes the form

$$V(r) = \begin{cases} -V_0, & r < b, \\ 0, & r \geq b, \end{cases} \qquad (2-47)$$

where b is the range of the nuclear force. For $< b$, i.e. inside the well, the radial equation is

$$\left[\frac{d^2}{\mathrm{d}r^2}+\frac{2}{r}\frac{d}{dr}-\frac{\ell(\ell+1)}{r^2}+K^2\right]R_\ell(\mathrm{K},r)=0, \qquad (2-48)$$

where $K^2=k^2+U_0$ and $U_0=\frac{2mV_0}{\hbar^2}$,

For $r < b$, the general solution of (2-48) is

$$R_\ell^I(\mathrm{K},r)=N_\ell(\mathrm{K})j_\ell(\mathrm{K}r) \;, \qquad (2-49)$$

where the superscript I refers to the internal region ($r < b$), and $N_\ell(\mathrm{K})$ is a normalization constant.

For $r > b$, i.e for the exterior region the radial equation is:

$$\left[\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}-\frac{\ell(\ell+1)}{r^2}+k^2\right]R_\ell(k,r)=0. \qquad (2-50)$$

which has the general solution

$$R_\ell^e(k,\mathrm{r})=B_\ell(k)[j_\ell(kr)-\tan\delta_\ell(k)n_\ell(kr)], \qquad (2-51)$$

where the superscript *e* refers to the external region. We use the boundary conditions that the wave function and its derivatives are continuous at r = b. This means that the internal and external solutions and their derivatives are continuous at the boundary $r = b$. Therefore we have,

$$N_\ell(\mathrm{K})j_\ell(\mathrm{Kb})=B_\ell(k)[j_\ell(k\mathrm{b})-\tan\delta_\ell(k)n_\ell(k\mathrm{b})]. \qquad (2-52)$$

and

$$\mathrm{K}N_\ell(\mathrm{K})j_\ell'(\mathrm{Kb})=B_\ell(k)[k\,j_\ell'(k\mathrm{b})-k\tan\delta_\ell(k)n_\ell'(k\mathrm{b})]. \qquad (2-53)$$

where $j_\ell'(\rho)$ means the derivative with respect to ρ.

To find the phase shifts $\delta_\ell(k)$ we divide (2-53) by (2-52) to eliminate the constants $N_\ell$and B_ℓ. Hence

$$\frac{Kj_\ell'(\mathrm{Kb})}{j_\ell(\mathrm{Kb})}=\frac{kj_\ell'(k\mathrm{b})-k\tan\delta_\ell(k)n_\ell'(k\mathrm{b})}{j_\ell(k\mathrm{b})-\tan\delta_\ell(k)n_\ell(k\mathrm{b})}. \qquad (2-54)$$

Solving for tan δ_ℓ we find

$$\tan \delta_\ell(k) = \frac{k j_\ell'(kb) j_\ell(Kb) - K j_\ell(kb) j_\ell'(Kb)}{k n_\ell'(kb) j_\ell(Kb) - K n_\ell(kb) j_\ell'(Kb)}. \qquad (2-55)$$

2-4 The Low energy limit:

The method of partial waves explained above is only useful when the number of partial waves contributing to the scattering cross section is small. This condition is achieved at low incident energies. By inspecting the Schrodinger equation in the internal region (equation (2 - 21)) it can be seen that when the incident energy is small compared to the depth of the potential, then the wave function in the internal region will not depend sensitively on E (Rodberg and Thaler, 1970). In addition, for finite range potentials as the incident energy decreases only a small number of partial waves contribute in the scattering process. For large partial waves, the reduced wave function in the vicinity of the origin behaves like $(kr)^{\ell+1}$ which means only the s-wave contributes to the scattering process at small energies. For $\ell = 0$ the corresponding spherical Bessel and Neumann functions are given by

$$j_0(kr) = \frac{\sin kr}{kr}, n_0(kr) = \frac{-\cos kr}{kr}. \qquad (2-56)$$

Hence, equation (2 - 52) results in the following expression for the s-wave phase shift:

$$\tan \delta_0 = \frac{k \tan Kb - K \tan kb}{K + k \tan k\, b \tan Kb}. \qquad (2-57)$$

Using equation (2-46), the corresponding total cross section takes the form

$$\sigma_{tot.} = \frac{4\pi}{k^2} \sin^2 \delta_0 = \frac{4\pi}{k^2} \frac{1}{1+\cot^2 \delta_0(k)}. \qquad (2-58)$$

As the incident energy increases, higher partial wave become progressively more important. The dependence of the phase shifts on energy as we move away from the low energy limit has been studied in the frame work of the effective range theory, which is briefly outlined in the following section.

2.5 Effective range theory:

In the previous section we discussed the theory of scattering in the zero energy limit, where the cross section is given by equation (2-58). Here, we shall illustrate the dependence of the s-wave scattering phase shifts as a function of the increasing incident energy. To this end consider an incident particle of energy E_1 and wave number

$k_1 = \frac{2mE_1}{\hbar^2}$. The corresponding reduced wave function $\upsilon_1(k,r)$ satisfies the Schrödinger equation

$$\upsilon_1'' + k_1^2\,\upsilon_1 - \mathrm{U(r)}\,\upsilon_1 = 0, \qquad (2-59)$$

where the prime denotes differentiation with respect to r and. Similarly, for another energy E_2 we have,

$$\upsilon_2'' + k_2^2\,\upsilon_2 - \mathrm{U(r)}\,\upsilon_2 = 0. \qquad (2-60)$$

Multiply equation (2-59) by υ_2 and (2-60) by υ_1 and subtracting leads to

$$\upsilon_2\upsilon_1'' - \upsilon_1\upsilon_2'' + k_1^2\upsilon_2\upsilon_1 - k_2^2\upsilon_2\upsilon_1 = 0. \qquad (2-61)$$

Upon integrating (2-61) from 0 to *B*, where *B* is arbitrary, we obtain

$$\upsilon_2\upsilon_1' - \upsilon_1\upsilon_2' \,|_0^B = (k_2^2 - k_1^2)\int_0^B \upsilon_2\upsilon_1\,\mathrm{dr}\,. \qquad (2-62)$$

When deriving the effective range expansion it is customary to choose the normalization of the wave functions such that in the asymptotic region we have (H. A. Bethe, 1949):

$$\upsilon_i = \frac{\sin(k_i\mathrm{r} + \delta_i)}{\sin\delta_i}\,, \qquad (2-63)$$

Now define free particle wave functions ψ_i, which are solutions of (2 - 59) and (2 - 60) for $U(r) = 0$, namely $\psi_i = \frac{\sin(k_i\mathrm{r}+\delta_i)}{\sin\delta_i}$. The sub-indices $i = 1, 2$ refer to the energies not the angular momenta. In addition, note that the form of ψ_i is valid for all *r* and

$$\lim_{r\to 0}\psi_i = 1 \qquad (2-64)$$

For ψ_i an equation similar to $(2-62)$ takes the form:

$$\psi_2\psi_1' - \psi_1\psi_2'|_0^B = (k_2^2 - k_1^2)\int_0^B \psi_1\psi_2 \, \mathrm{dr} \,. \qquad (2-65)$$

By subtracting (2-65) from (2-62) we get

$$\upsilon_2\upsilon_1' - \upsilon_1\upsilon_2' - \psi_2\psi_1' + \psi_1\psi_2'|_0^B$$
$$= (k_2^2 - k_1^2)\int_0^B (\upsilon_2\upsilon_1 - \psi_1\psi_2)\, dr. \qquad (2-66)$$

If the upper limit *B* is chosen large compared to the range of the nuclear forces, then for $r > B$ each function υ_i will be identical to the corresponding ψ_i. Hence, the contribution of the left hand side coming from evaluating the upper limits $r = B$ vanishes. Further, in the asymptotic region (outside the range of the potential) the contribution of the integral on the right hand side is zero. Consequently, extending the upper limit *B* to ∞ and using the boundary condition $\upsilon_1(0) = \upsilon_2(0) = 0$ then equation (2-66) reduces to

$$\psi_1\psi_2' - \psi_2\psi_1' \Big|_{\mathrm{r}=0} = (k_2^2 - k_1^2)\int_0^\infty (\psi_1\psi_2 - \upsilon_1\upsilon_2)\, dr. \qquad (2-67)$$

From equation (2-63),

$$\psi_{1,2} = \frac{\sin(k_{1,2}\mathrm{r} + \delta_{1,2})}{\sin\delta_{1,2}}, \qquad (2-68)$$

hence

$$\psi_{1,2}' = \frac{k_{1,2}\cos(\mathrm{k}_{1,2}\mathrm{r} + \delta_{1,2})}{\sin\delta_{1,2}} \,. \qquad (2-69)$$

Substituting $\psi_{1,2}$ and $\psi_{1,2}'$ in equation (2-67) we obtain

$$k_2\cot\delta_2 - k_1\cot\delta_1 = (k_2^2 - k_1^2)\int_0^\infty (\psi_1\psi_2 - \upsilon_1\upsilon_2)\, \mathrm{dr}. \qquad (2-70)$$

For the special case $k_1 = 0$, we may use

$$\lim_{k_1\to 0} k_1\cot\delta_1 = -\frac{1}{a}, \qquad (2-71)$$

where a is the scattering length and has the dimension of length and is defined in equation (2 - 75) below.

In equation (2-70) if we drop the subscript 2 then the equation can be written as

$$k\cot\delta = -\frac{1}{a} + k^2 \int_0^\infty (\psi\,\psi_0 - \upsilon\,\upsilon_0)\mathrm{dr}. \qquad (2-72)$$

Defining $r_{\text{eff}} = 2\int_0^\infty (\psi\,\psi_0 - \upsilon\,\upsilon_0)\,\mathrm{dr}$ reduces equation (2-72) to

$$k\cot\delta = -\frac{1}{a} + \frac{1}{2}k^2 r_{\text{eff}}\,. \qquad (2-73)$$

The quantity r_{eff} is called effective range and has a dimension of length.

Using equation (2-73) we can express (2-58) as

$$\sigma = \frac{4\pi}{k^2}\frac{1}{1+\cot^2\delta_\circ} = \frac{4\pi\, a^2}{a^2k^2 + \left(1 - \frac{1}{2}a\ r_{\text{eff}}\,k^2\right)^2}\ . \qquad (2-74)$$

Where the influence of the potential is represented by the effective range r_{eff} and the scattering length *a*. So, at low energies, the cross section is not affected by the details of the form of the potential since with any other reasonable it will always be possible to adjust the depth and the range in such a way as to reproduce the value of *a* and r_{eff} (Bertulani, 2005).

The scattering length has a dimension of length. Mathematically it is defined as

$$a = -\lim_{k\to 0}\frac{\sin\delta_0}{k} \qquad (2-75)$$

We can visualize the scattering length to be the intercept of the asymptotic wave function in the zero energy limit with the *r*-axis. For a repulsive potential the scattering length is positive as seen in Figure (2-3-a). However, for an attractive potential, the intercept can be on the negative side which means that *a* is negative and there is no bound state, as shown in Figure (2-3-b). However, If we increase the attraction, the

outside-wave function can again cross the *r*-axis on the positive side and hence *a* is positive signifying a bound state Figure (3-c), (Sakurai, 1994) (Gasiorowicz, 1974).

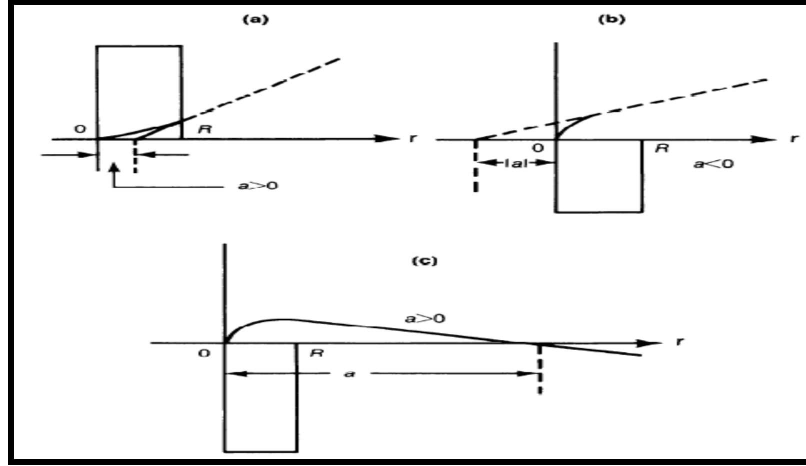


Figure (2-2): Plot of υ(r) versus r for (a) repulsive potential. (b) attractive potential, and(c) deeper attraction. The intercept *a* of the zero energy outside wave function with the *r axis* is shown for each three cases.

Chapter Three

Formalism

3-1 Introduction

In this chapter we shall derive the perturbation formulas for the changes in the s-wave functions and the corresponding changes in the phase shifts when small perturbations in the incident energy are introduced. We shall assume the nuclear potential to be velocity-dependent.

Kisslinger developed a velocity-dependent potential that was successful in describing the scattering of mesons off complex nuclei, which predicted the predominantly p-wave nature of the elementary pion-nucleon coherent scattering. Further, in nuclear physics a model assuming the nucleon-nucleon interaction to be velocity dependent reproduced the ^1S, ^1D, and ^1G singlet-even phase shifts without having to use a hard core potential in the vicinity of the origin (Kisslinger, 1955).

The velocity-dependent potential is expressed as:

$$\nabla.(\rho(r)\nabla\psi(\vec{r})) = \rho(r)\nabla^2\psi(\vec{r}) + \nabla\rho(r).\nabla\psi(\vec{r}). \qquad (3-1)$$

where $\rho(r)$ is an isotropic function of the radial variable r and $\psi(\vec{r})$ is the wave function in three dimensions. Generally, $\rho(r)$ can be taken to represent the spatially varying nuclear density. Consequently, the second term on the right is sensitive to the diffuse edge in nuclei. Further, the first term is proportional to the kinetic energy and combines with the kinetic-energy term in the Schrödinger equation hence the name velocity-dependent potential.

If we substitute equation (3-1) into equation (2-2) then we get Schrödinger's equation which includes the Kisslinger term

$$\left[\frac{-\hbar^2}{2m}\{\nabla^2 - \rho(r)\nabla^2 - \nabla\rho(r).\nabla\} + V(r)\right]\psi(\vec{r}) = E\psi(\vec{r}). \qquad (3-2)$$

Multiplying the last equation by $\frac{-2m}{\hbar^2}$ leads to

$$\left[\nabla^2 - \rho(r)\nabla^2 - \nabla\rho(r).\nabla - \frac{2m}{\hbar^2}V(r)\right]\psi(\vec{r}) = -\frac{2m}{\hbar^2}E\psi(\vec{r}). \qquad (3-3)$$

Inserting (2-17) and (2-18) into (3-3) gives

$$\left[(1-\rho(r))\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)-\frac{L^2}{\hbar^2 r^2}\right)-\frac{\partial\rho(r)}{\partial r}\frac{\partial}{\partial r}-\frac{2m}{\hbar^2}V(r)\right]\psi(\vec{r})$$
$$=-\frac{2m}{\hbar^2}\mathrm{E}\,\psi(\vec{r}). \qquad (3-4)$$

Substituting (2-20) into the last equation and using $L^2 Y_\ell^m(\theta,\phi)=\ell(\ell+1)\hbar^2 Y_\ell^m(\theta,\phi)$, we obtain the radial Schrödinger equation for a velocity-dependent potential; namely

$$(1-\rho(r))\left[\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}\right]R_\ell(k,r)-\frac{(1-\rho(r))\ell(\ell+1)R_\ell(k,r)}{r^2}-\frac{d\rho(r)}{dr}\frac{dR_\ell(k,r)}{dr}$$
$$+\frac{2m}{\hbar^2}[(E-V(r)]R_\ell(k,r)=0 \qquad (3-5)$$

For simplicity, we shall consider the s-wave case, ℓ =0, hence

$$(1-\rho(r))\left[\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}\right]R_o(k,r)+\left[\frac{2m}{\hbar^2}\mathrm{E}-\frac{d\rho(r)}{dr}\frac{d}{dr}-\frac{2m}{\hbar^2}\mathrm{V}(r)\right]R_o(k,r)$$
$$=0, \qquad (3-6)$$

where $R_o(k,r)$ is the radial part of the wave function in the s-wave case.

In terms of the reduced wave function $v_l(k,r)=rR_\ell(k,r)$, we can write the s-wave Schrödinger equation with a velocity dependent term as:

$$(1-\rho(r))v''(k,\mathrm{r})-\left[v'(k,\mathrm{r})-\frac{v(k,r)}{r}\right]\rho'(r)=\frac{2m}{\hbar^2}[V(r)-E]v(k,r), \qquad (3-7)$$

where the prime denotes differentiation with respect to the radial variable r.

3-2 First-order wave function correction in the asymptotic region

In this section we shall investigate the energy dependence of the wave function when the incident energy is changed from its unperturbed value E_0 by a small amount ΔE. For simplicity, the local and velocity-dependent parts of the potential are assumed to have a given radius *b* beyond which both identically vanish. In the asymptotic region the corresponding solution of (3-7) is

$$\tilde{v}\,(k,\mathrm{r})=\frac{1}{k}\sin(kr+\delta). \qquad (3-8)$$

It is customary when deriving the effective range expansion to choose a normalization such that, for distances larger than b, the wave functions behave as

$$\upsilon\,(k,\mathrm{r}) = \frac{k}{\sin\delta}\tilde{\upsilon}(k,\mathrm{r}) \qquad (3-9)$$

Hence for $\ell = 0$,

$$\upsilon(k,r) = \frac{\sin(k\mathrm{r}+\delta)}{\sin\delta}, \qquad (3-10)$$

where δ is the exact s-wave phase shift.

In order to derive the wave function changes upon the introduction of a small change in energy ΔE, we start by expanding the s-wave function as a power series in k^2 about some value k_o^2. Hence

$$\upsilon(k,r) = \sum_{n=0}^{\infty}\left(\frac{\hbar^2}{2m}\right)^n \frac{(k^2-k_o^2)^n}{n!}\upsilon_n(k_o,r) = \sum_{n=0}^{\infty}\frac{(\Delta \mathrm{E})^n}{n!}\,\upsilon_n(k_o,r), \qquad (3-11)$$

where the perturbed energy E is given by

$$E = E_o + \Delta E \qquad (3-12)$$

Clearly, the large r-dependence of the first order change in $\upsilon(k,r)$ is fixed by differentiating (3-11) with respect to k^2 then setting $k = k_o$. Using the expression for $\upsilon(k,r)$ given in (3-10) we obtain:

$$\upsilon_1(k_o,r) = \frac{2m}{\hbar^2}\frac{d}{dk^2}\left(\frac{\sin(k\,r+\delta)}{\sin\delta}\right)\Bigg|_{k^2=k_o^2} \qquad \text{for } r \geq b \qquad (3-13)$$

$$\upsilon_1(k_o,r) = \frac{2m}{\hbar^2}\left(\frac{r\cos(k_o r+\delta_o)}{2k_o\sin\delta_o} - \frac{\delta'\,\sin k_o r}{2k_o\,\sin^2\delta_o}\right), \qquad (3-14)$$

where $\delta' = \frac{d\delta}{dk}$ is evaluated at $(k = k_o)$ and the subscript (o) denotes the unperturbed case i. e, ΔE = 0.

3-3 First-order wave function correction:

In this section we shall derive the first-order wave function correction when the incident energy is perturbed by a small amount ΔE from the initial value E_o.

Inserting (3-11) and (3-12) into (3-7) we obtain

$$(1-\rho(r))\left(\upsilon_o''(r)+\Delta E\upsilon_1''(r)+\cdots+\frac{\Delta E^n}{n!}\upsilon_n''(r)\right)-\left(\upsilon_o'(r)+\Delta E\upsilon_1'(r)+\cdots+\frac{\Delta E^n}{n!}\upsilon_n'(r)-\frac{\upsilon_o(r)}{r}-\frac{\Delta E\ \upsilon_1(r)}{r}-\cdots-\frac{\Delta E^n\upsilon_n(r)}{n!\,r}\right)\rho'(r)=\frac{2m}{\hbar^2}(V(r)-E_o-\Delta E)\left(\upsilon_o(r)+\Delta E\upsilon_1(r)+\cdots+\frac{\Delta E^n}{n!}\upsilon_n(r)\right), \qquad (3-15)$$

where the dependence of the wave function on k_o is not shown for clarity of presentation.

By comparing the coefficients of $(\Delta E)^o$ on both sides, we obtain the unperturbed Schrödinger equation corresponding to $\Delta E = 0$ (perturbation is switched off);

$$(1-\rho(r))\upsilon_o''(r)-\left(\upsilon_o'(r)-\frac{\upsilon_o(r)}{r}\right)\rho'(r)=\frac{2m}{\hbar^2}(V(r)-E_o)\upsilon_o(r), \qquad (3-16)$$

where $\upsilon_0(r)$ is the unperturbed wave function.

Further, from equation (3-15) we find that the coefficient of ΔE^n, where $n \geq 1$, satisfies the following equation

$$(1-\rho(r))\frac{\upsilon_n''(r)}{n!}-\left(\frac{\upsilon_n'(r)}{n!}-\frac{\upsilon_n(r)}{n!\,r}\right)\rho'(r)=\frac{2m}{\hbar^2}(V(r)-E_o)\frac{\upsilon_n(r)}{n!}-\frac{2m}{\hbar^2}\frac{\upsilon_{n-1}(r)}{(n-1)!}\ . \qquad (3-17)$$

To find the first-order wave function correction we set n=1, which leads to

$$(1-\rho(r))\upsilon_1''(r)-\left[\upsilon_1'(r)-\frac{\upsilon_1(r)}{r}\right]\rho'(r)=-\frac{2m}{\hbar^2}\upsilon_o(r)+\frac{2m}{\hbar^2}[V(r)-E_o]\upsilon_1(r) \qquad (3-18)$$

Evaluating for $\frac{2m}{\hbar^2}[V(r)-E_o]$ in equation (3-18) using the unperturbed equation (3-16) leads to

$$\left(1-\rho(r)\right)\upsilon_1''(r)-\left[\upsilon_1'(r)-\frac{\upsilon_1(r)}{r}\right]\rho'(r)=-\frac{2m}{\hbar^2}\upsilon_o(r)$$

$$+\upsilon_1(r)\left(1-\rho(r)\right)\frac{\upsilon_o''(r)}{\upsilon_o(r)}-\left[\upsilon_o'(r)-\frac{\upsilon_o(r)}{r}\right]\frac{\rho'(r)\upsilon_1(r)}{\upsilon_o(r)}. \qquad (3-19)$$

Multiplying (3-19) by $\upsilon_o(r)$ and rearranging results in

$$\left(1-\rho(r)\right)[\upsilon_o(r)\upsilon_1''(r)-\upsilon_1(r)\upsilon_o''(r)]+[\upsilon_o'(r)\upsilon_1(r)-\upsilon_o(r)\upsilon_1'(r)]\rho'(r)$$
$$=-\frac{2m}{\hbar^2}\upsilon_o^2(r). \qquad (3-20)$$

The left hand side is an exact derivative, hence we may write

$$\frac{d}{dr}\left[(1-\rho(r))\left(\upsilon_1'(r)\upsilon_o(r)-\upsilon_1(r)\upsilon_o'(r)\right)\right]=-\frac{2m}{\hbar^2}\upsilon_o^2(r). \qquad (3-21)$$

Integrating the last equation from the origin to *r* and using the condition $\upsilon_n(0)=0$ for $n \geq 0$ leads to:

$$(1-\rho(r))[\upsilon_1'(r)\upsilon_o(r)-\upsilon_1(r)\upsilon_o'(r)]=-\frac{2m}{\hbar^2}\int_0^r \upsilon_o^2(r)\,dr. \qquad (3-22)$$

We can write (3-22) after some algebraic manipulation as

$$\upsilon_o^2(r)\frac{d}{dr}\left(\frac{\upsilon_1(r)}{\upsilon_o(r)}\right)=-\frac{2m}{\hbar^2}\frac{1}{(1-\rho(r))}\int_0^r \upsilon_o^2(r)dr. \qquad (3-23)$$

Integrating both sides from $r = b$ to r gives

$$\frac{\upsilon_1(r)}{\upsilon_o(r)}=-\frac{2m}{\hbar^2}\int_b^r\frac{dr'}{(1-\rho(r'))\upsilon_o^2(r')}\int_0^{r'}\upsilon_o^2(r'')dr''+C_1. \qquad (3-24)$$

Equivalently, we may write

$$\upsilon_1(r)=-\frac{2m}{\hbar^2}\upsilon_o(r)\int_b^r\frac{dr'}{(1-\rho(r'))\upsilon_o^2(r')}\int_0^{r'}\upsilon_o^2(r'')dr''$$
$$+C_1\upsilon_o(r), \qquad (3-25)$$

where C_1 is the constant of integration given by

$$C_1=\left.\frac{\upsilon_1(r)}{\upsilon_o(r)}\right|_{r=b}. \qquad (3-26)$$

3-4 Second-order wave function correction:

To determine the expression for the second-order wave function correction we start by setting *n=2* in equation (3-17), which results in

$$\left(1-\rho(r)\right)\frac{\upsilon_2''(r)}{2}-\left(\frac{\upsilon_2'(r)}{2}-\frac{\upsilon_2(r)}{2r}\right)\rho'(r) = \frac{2m}{\hbar^2}\left(V(r)-E_o\right)\frac{\upsilon_2(r)}{2}-\frac{2m}{\hbar^2}\upsilon_1(r). \qquad (3-27)$$

Substituting for the factor $\frac{2m}{\hbar^2}[V(r)-E_o]$ using the unperturbed Schrödinger equation given in (3-16) we get

$$\left(1-\rho(r)\right)\upsilon_2''(r)-\upsilon_2'(r)\rho'(r)+\frac{\upsilon_2(r)\rho'(r)}{r} = \left(1-\rho(r)\right)\frac{\upsilon_\circ''(r)\upsilon_2(r)}{\upsilon_0(r)}-\left(\upsilon_\circ'(r)-\frac{\upsilon_\circ(r)}{r}\right)\frac{\rho'(r)\upsilon_2(r)}{\upsilon_\circ(r)} - 2\frac{2m}{\hbar^2}\ \upsilon_1(r). \qquad (3-28)$$

Multiplying both sides in (3-28) by $\upsilon_0(r)$ and rearranging the terms we get,

$$\left(1-\rho(r)\right)[\upsilon_o(r)\upsilon_2''(r)-\upsilon_2(r)\upsilon_o''(r)]+[\upsilon_o'(r)\upsilon_2(r)-\upsilon_o(r)\upsilon_2'(r)]\rho'(r) = -2\frac{2m}{\hbar^2}\ \upsilon_o(r)\upsilon_1(r). \qquad (3-29)$$

The left hand side term is an exact derivative, hence:

$$\frac{d}{dr}\left[\left(1-\rho(r)\right)\left(\upsilon_o(r)\upsilon_2'(r)-\upsilon_2(r)\upsilon_o'(r)\right)\right]=-2\frac{2m}{\hbar^2}\ \upsilon_o(r)\upsilon_1(r). \qquad (3-30)$$

Integrating the last equation from 0 to r leads to

$$\left[\left(1-\rho(r)\right)\left(\upsilon_o(r)\upsilon_2'(r)-\upsilon_2(r)\upsilon_o'(r)\right)\right]=-2\frac{2m}{\hbar^2}\int_0^r \upsilon_o(r)\upsilon_1(r)dr. \qquad (3-31)$$

We can write the above equation as

$$\upsilon_o^2(r)\frac{d}{dr}\left(\frac{\upsilon_2(r)}{\upsilon_\circ(r)}\right)=-2\left(\frac{2m}{\hbar^2}\right)\frac{1}{1-\rho(r)}\int_0^r \upsilon_\circ(r)\upsilon_1(r)dr. \qquad (3-32)$$

The second-order wave function correction is obtained by integrating the last equation from *b* to *r*, which finally leads to

$$v_2(r) = -2\left(\frac{2m}{\hbar^2}\right)v_{\circ}(r)\int_b^r \frac{dr'}{\left(1-\rho(r')\right)v_o^2(r')}\int_0^{r'} v_o\,(r'')v_1(r'')dr'' + C_2\, v_o(r). \qquad (3-33)$$

where the constant C_2 is given by

$$C_2 = \left.\frac{v_2(r)}{v_o(r)}\right|_{r=b}. \qquad (3-34)$$

3-5 Third-order wave function correction:

To determine the expression for the third-order correction to wave function we start by setting n =3 in equation (3-17), which gives

$$\left(1-\rho(r)\right)\frac{v_3''(r)}{3!} - \left(\frac{v_3'(r)}{3!} - \frac{v_3(r)}{3!\,r}\right)\rho'(r) = \frac{2m}{\hbar^2}\left(V(r)-E_o\right)\frac{v_3(r)}{3!} - \frac{2m}{\hbar^2}\frac{v_2(r)}{2}. \qquad (3-35)$$

Using the unperturbed Schrödinger equation (3-16) to evaluate for $\frac{2m}{\hbar^2}[V(r)-E_o]$ gives

$$\left(1-\rho(r)\right)v_3''(r) - v_3'(r)\rho'(r) = \left(1-\rho(r)\right)\frac{v_o''(r)v_3(r)}{v_o(r)} - \frac{v_o'(r)\rho'(r)v_3(r)}{v_o(r)} = -3\frac{2m}{\hbar^2}v_2(r). \qquad (3-36)$$

Multiplying both sides in (3-36) by $v_o(r)$ and rearranging results in

$$\frac{d}{dr}\left[\left(1-\rho(r)\right)\left(v_o(r)v_3'(r) - v_3(r)v_o'(r)\right)\right] = -3\frac{2m}{\hbar^2}\, v_o(r)v_2(r). \qquad (3-37)$$

Integrating the last equation from 0 to r

$$v_o^2(r)\frac{d}{dr}\left(\frac{v_3(r)}{v_o(r)}\right) = -3\left(\frac{2m}{\hbar^2}\right)\frac{1}{\left(1-\rho(r')\right)}\int_0^r v_o(r')v_2(r')dr'. \qquad (3-38)$$

By integrating the last equation from b to r the final expression for $v_3(r)$ is expressed as

$$v_3(r) = -3\left(\frac{2m}{\hbar^2}\right)v_o(r)\int_b^r \frac{dr'}{\left(1-\rho(r')\right)v_o^2(r')}\int_0^{r'} v_o\,(r'')v_2(r'')dr'' + C_3\, v_o(r). \qquad (3-39)$$

where C_3 is the constant of integration and is given by

$$C_3 = \frac{\upsilon_3(r)}{\upsilon_o(r)}\Big|_{r=b}. \qquad (3-40)$$

By inspecting the first, second and third-order wave function corrections we can express the n^{-th}-order wave function correction as

$$\upsilon_n(r) = -n\left(\frac{2m}{\hbar^2}\right)\upsilon_o(r)\int_b^r \frac{dr'}{(1-\ \rho(r')\ )\upsilon_o^2(r')}\int_0^{r'} \upsilon_o\ (r'')\upsilon_{n-1}(r'')dr'' + C_n\,\upsilon_o(r), \qquad (3-41)$$

where C_n is the constant of the integration given by

$$C_n = \frac{\upsilon_n(r)}{\upsilon_o(r)}\Big|_{r=b}. \qquad (3-42)$$

The wave function corrections $\upsilon_n(k_0,r)$ are finite at the zeros of the unperturbed functions $\upsilon_0(k_0,r)$ as shown by (M. Jaghoub, 2006).

3-6 Energy dependence of the scattering phase shifts

In order to study the variation of the phase shifts as a function of a small change in the incident energy we shall start by expanding $k\ \cot\delta$ as a power series about a point $k^2 = k_o^2$, namely

$$\alpha(k) = k\ \cot\delta = \alpha_o + \frac{\hbar^2}{2m}(k^2 - k_o^2)\alpha_1 + \left(\frac{\hbar^2}{2m}\right)^2\frac{(k^2-k_o^2)^2\alpha_2}{2!} + \cdots\ , \qquad (3-43)$$

where δ is the perturbed phase shift. In the limit $k_o \longrightarrow 0$, α_o and α_1 are the conventional scattering length and effective range respectively . Clearly, the α_n terms of the series are given by

$$\alpha_n = \left(\frac{2m}{\hbar^2}\right)^n\left(\frac{\partial}{\partial k^2}\right)^n k\ \cot\delta\Big|_{k=k_o}. \qquad (3-44)$$

3-6.1 Determining the effective range parameter α_1

Setting *n=1* in the last equation results in the following expressions for α_1,

$$\alpha_1 = \frac{2m}{\hbar^2}\left[\frac{\cot\delta_o}{2k} - \frac{\delta'}{2\ \sin^2\delta_o}\right] \qquad (3-45)$$

where $\delta' = \frac{d\delta}{dk}$ is evaluated at $k = k_o$. In order to find α_1 we clearly need to determine δ'. To achieve this we shall start by rewriting (3-7) for a given energy $E_1 = \frac{\hbar^2 k_1^2}{2m}$ as

$$(1 - \rho(r))\upsilon_1''(r) - \left[\upsilon_1'(r) - \frac{\upsilon_1(r)}{r}\right]\rho'(r) + k_1^2\upsilon_1(r) - \frac{2m}{\hbar^2} V(r)\upsilon_1(r) = 0 . \quad (3-46)$$

For another energy $E_2 = \frac{\hbar^2 k_2^2}{2m}$ the corresponding equation is

$$(1 - \rho(r))\upsilon_2''(r) - \left[\upsilon_2'(r) - \frac{\upsilon_2(r)}{r}\right]\rho'(r) + k_2^2\upsilon_2(r) - \frac{2m}{\hbar^2} V(r)\upsilon_2(r) = 0. \quad (3-47)$$

Multiply (3-46) by $\upsilon_2(r)$ and (3-47) by $\upsilon_1(r)$ and subtracting leads to

$$(1 - \rho(r))[\upsilon_1''(r)\upsilon_2(r) - \upsilon_2''(r)\upsilon_1(r)] - \left[\upsilon_1'(r) - \frac{\upsilon_1(r)}{r}\right]\rho'(r)\upsilon_2(r) + \left[\upsilon_2'(r) - \frac{\upsilon_2(r)}{r}\right]\rho'(r)\upsilon_1(r) + (k_1^2 - k_2^2)\upsilon_2(r)\upsilon_1(r) = 0. \quad (3-48)$$

After some algebraic manipulation we can write (3-48) as:

$$(1 - \rho(r))[\upsilon_1''(r)\upsilon_2(r) - \upsilon_2''(r)\upsilon_1(r)] + [\upsilon_2'(r)\upsilon_1(r) - \upsilon_1'(r)\upsilon_2(r)]\rho'(r) + (k_1^2 - k_2^2)\upsilon_2(r)\upsilon_1(r) = 0. \quad (3-49)$$

Obviously the left hand side term is an exact derivative, hence,

$$\frac{d}{dr}[(1 - \rho(r))[\upsilon_1'(r)\upsilon_2(r) - \upsilon_2'(r)\upsilon_1(r)]] + (k_1^2 - k_2^2)\upsilon_2(r)\upsilon_1(r) = 0. \quad (3-50)$$

Integrate the last equation from 0 to b, where b is the range of the potential, and using the condition $\upsilon_n(0) = 0$ for $n \geq 0$ leads to:

$$(1 - \rho(r))[\upsilon_1'(r)\upsilon_2(r) - \upsilon_2'(r)\upsilon_1(r)]\Big|_{r=b} + \int_0^b (k_1^2 - k_2^2)\upsilon_2(r)\upsilon_1(r)dr = 0. \quad (3-51)$$

Since $\rho(b) = 0$ (ρ(r) is assumed to vanish for $r \geq b$) then the above equation becomes

$$[\upsilon_2(r)\,\upsilon_1'(r) - \upsilon_1(r)\upsilon_2'(r)]\Big|_{r=b} + \int_0^b (k_1^2 - k_2^2)\upsilon_2(r)\upsilon_1(r)dr = 0. \quad (3-52)$$

Differentiate (3-52) with respect to k_2 results in

$$\upsilon_1'(r)\frac{\partial \upsilon_2(r)}{\partial k_2} - \upsilon_1(r)\frac{\partial \upsilon_2'(r)}{\partial k_2}]\Big|_{r=b} + \int_0^b (0-2k_2)\upsilon_2(r)\upsilon_1(r)dr + \int_0^b (k_1^2-k_2^2)\frac{\partial \upsilon_2(r)}{\partial k_2}\upsilon_1(r) = 0. (3-53)$$

In the limit $k_2 \to k_1 = k_0$, we have $\upsilon_2(\mathrm{r}) \to \upsilon_1(\mathrm{r})$ and (3-53) reduces to

$$\upsilon_o'(r)\frac{\partial \upsilon_o(r)}{\partial k_0} - \upsilon_o(r)\frac{\partial \upsilon_o'(r)}{\partial k_0}\Big|_{r=b} = \int_0^b 2k_0\, \upsilon_o^2(r) \qquad (3-54)$$

Or,

$$\frac{1}{2k_0}\left[\upsilon_o'(r)\frac{\partial \upsilon_o(r)}{\partial k_o} - \upsilon_o(r)\frac{\partial \upsilon_o'(r)}{\partial k_o}\right]\Big|_{r=b} = \int_0^b \upsilon_o^2(r)dr, \qquad (3-55)$$

where the prime denotes differentiation with respect to the radial variable r.

To find an expression for δ' we start from (3-10) and find $\upsilon_o'(r)$ by differentiating (3-10) with respect to r then substitute for $\frac{\partial \upsilon_0(r)}{\partial k_0}$ and $\frac{\partial \upsilon_0'(r)}{\partial k_0}$ in (3-55), where $\upsilon_o(r)$ is the unperturbed wave function and δ_o is the unperturbed phase shift. We obtain,

$$\upsilon_o'(\mathrm{r}) = \frac{\partial \upsilon_o(r)}{\partial r} = \frac{k_o \cos(k_o \mathrm{r} + \delta_o)}{sin\delta_o}, \qquad (3-56)$$

and

$$\frac{\partial \upsilon_o(r)}{\partial k_0} = \frac{\partial}{\partial k_o}\left[\frac{sin(k_o r + \delta_o)}{sin\delta_o}\right] = \frac{-\delta' cos\delta_o sin(k_o r + \delta_o)}{sin^2\delta_o} + \frac{\cos(k_o r + \delta_o)}{sin\delta_o}(r + \delta'). \qquad (3-57)$$

Now differentiating (3-56) with respect to k_o leads to

$$\frac{\partial \upsilon_o'(\mathrm{r})}{\partial k_o} = \frac{\cos(k_o r + \delta_o)}{sin\delta_o} - \frac{k_o \cos(k_o r + \delta_o)\ \cos\delta_o}{sin^2\delta_o}\delta' - \frac{k_o \sin(k_o r + \delta_o)}{sin\delta_o}(r + \delta'). \qquad (3-58)$$

To find an expression for the phase shift changes due to the perturbing energy ΔE , we insert (3-56), (3-57) and (3-58) into (3-55) to get

$$\frac{1}{2k_0}\left[\frac{k_o\cos(k_or+\delta_o)}{\sin\delta_o}\left(\frac{-\delta'\cos\delta_o sin(k_or+\delta_o)}{sin^2\delta_o}+\frac{\cos(k_or+\delta_o)}{sin\delta_o}(r+\delta')\right)\right.$$
$$-\frac{sin(k_or+\delta_o)}{sin\delta_o}\left(\frac{\cos(k_or+\delta_o)}{sin\,\delta_o}-\frac{k_o\cos(k_or+\delta_o)\cos\delta_o}{sin^2\delta_o}\delta'\right.$$
$$\left.\left.-\frac{k_o\sin(k_or+\delta_o)}{sin\delta_o}(r+\delta')\right)\right]\Bigg|_{r=b}=\int_0^b \upsilon_o^2(r)dr. \qquad (3-59)$$

Using trigonometric manipulation, equation (3-59) simplifies to

$$\frac{k_ob}{sin^2\delta_o}+\frac{k_o\,\delta'}{sin^2\delta_o}-\frac{sin\,(k_ob+\delta_o)cos(k_ob+\delta_o)}{sin^2\delta_o}=2k_o\int_0^b \upsilon_o^2(r)dr. \qquad (3-60)$$

Multiply the last equation by $(\sin^2\delta_0)/\,k_o$ and use the identity sin2θ =2 sinθ cosθ then

(3-60) reduces to

$$\delta'=2\,sin^2\delta_o\int_0^b \upsilon_o^2(r)dr+\frac{sin\,(2k_ob+2\delta_o)}{2k_o}-b. \qquad (3-61)$$

Where as we mentioned earlier $\delta'=\frac{d\delta}{dk}$ is evaluated at $k=k_0$ and the subscript (0)

denotes the unperturbed case i.e. $k = k_0$.

Substituting (3-61) in (3-45) gives

$$\alpha_1=\frac{2m}{\hbar^2}\left[\frac{cot\,\delta_o}{2k_o}-\left(\int_0^b \upsilon_o^2(r)dr+\frac{sin(2k_ob+2\delta_o)}{4k_o sin^2\delta_o}-\frac{b}{2\,sin^2\delta_o}\right)\right]. \qquad (3-62)$$

Defining $\upsilon_o^f(r)=\frac{sin\,(k_or+\delta_o)}{\sin\,\delta_o}$ we have

$$\int_0^b(\upsilon_o^f(r))^2dr=\int_0^b\frac{\sin^2\,(k_or+\delta_o)}{\sin^2\delta_o}dr$$
$$=\frac{cot\,\delta_o}{2k_o}-\frac{\sin(2k_ob+2\delta_o)}{4k_o\sin^2\delta_o}+\frac{b}{2\sin^2\delta_o}. \qquad (3-63)$$

Substituting (3-63) in (3-62) we get

$$\alpha_1=\frac{2m}{\hbar^2}\left[\int_0^b(\upsilon_o^f(r))^2dr-\int_0^b \upsilon_o^2(r)dr\right], \qquad (3-64)$$

Since $\upsilon_o^f(r)$ coincides with $\upsilon_o(r)$ for $r \geq b$, we may extend the upper limit in the last equation to infinity, hence

$$\alpha_1 = \frac{2m}{\hbar^2}\left[\int_0^{\infty} (\upsilon_o^f(r))^2 dr - \int_0^{\infty} \upsilon_o^2(r) dr\right], \qquad (3-65)$$

In the zero energy limit the above expression for α_1 coincides with Bethe's formula for the effective range. Since $\upsilon_o^f(r)$ and $\upsilon_o(r)$ are identical in the asymptotic region then α_1 may be viewed as the difference in the areas under the wave functions corresponding to the free and interacting wave functions in the interaction region.

3-6.2 Determining the shape parameter α_2

To find α_2 we start from (3-30) and integrate it from 0 to b, to get

$$(1-\rho(r))\left[\left[\upsilon_o(r)\,\upsilon_2'(r) - \upsilon_2(r)\upsilon_o'(r)\right]\right]\Big|_{r\,=\,b} = -2\frac{2m}{\hbar^2}\int_0^b \upsilon_o(r)\upsilon_1(r)dr. \qquad (3-66)$$

Since $\rho(b) = 0$ then the above equation becomes

$$\left[\upsilon_o(r)\,\upsilon_2'(r) - \upsilon_2(r)\upsilon_o'(r)\right]\Big|_{r\,=\,b} = -2\frac{2m}{\hbar^2}\int_0^b \upsilon_o(r)\upsilon_1(r)dr. \qquad (3-67)$$

By inspecting (3-44), knowledge of α_2 requires the determination of $\delta'' = \frac{d^2\delta}{d^2k}$ evaluated at $= k_0$. In order to achieve this we note that

$$\upsilon_2(r) = \frac{2m}{\hbar^2}\frac{d}{dk_0^2}\left[\upsilon_1(r)\right]$$
$$= \left(\frac{2m}{\hbar^2}\right)^2\left[-\frac{r\cos(k_o r+\delta_o)}{4k_o^3\sin\delta_o} - \frac{r^2\sin(k_o r+\delta_o)}{4k_o^2\sin\delta_o} - \frac{\delta' r\cos(k_o r)}{2k_o^2\sin^2\delta_o}\right.$$
$$\left. + \frac{\delta'\sin(k_o r)}{4k_o^3\sin^2\delta_o} + \frac{\delta'^2\cot\delta_o\sin(k_o r)}{2k_o^2\sin^2\delta_o} - \frac{\delta''\sin(k_o r)}{4k_o^2\sin^2\delta_o}\right], \qquad (3-68)$$

and

$$\upsilon_2'(r) = \frac{d}{dr}[\upsilon_2(r)]$$
$$= \left(\frac{2m}{\hbar^2}\right)^2 \left[-\frac{\cos(k_o r+\delta_o)}{4k_o^3\sin\delta_o} - \frac{r^2\cos(k_o r+\delta_o)}{4k_o\sin\delta_o} - \frac{r\sin(k_o r+\delta_o)}{4k_o^2\sin\delta_o} - \frac{\delta'\cos(k_o r)}{4k_o^2\sin^2\delta_o} + \frac{r\delta'\sin(k_o r)}{2k_o\sin^2\delta_o} + \frac{\delta'^2\cos(k_o r)\cot\delta_o}{2k_o\sin^2\delta_o} - \frac{\delta''\cos(k_o r)}{4k_o\sin^2\delta_o}\right]. \quad (3-69)$$

Substitute (3-68), and (3-69) into (3-67) we get

$$2\left(\frac{2m}{\hbar^2}\right)\int_0^b \upsilon_0(r)\upsilon_1(r)dr = \left(\frac{2m}{\hbar^2}\right)^3 \left[-\frac{b\cos(2bk_o+2\delta_o)}{4k_o^2\sin^2\delta_o} + \frac{\sin(2bk_o+2\delta_o)}{8k_o^3\sin^2\delta_o} - \frac{\delta' b\cot\delta_o}{2k_o\sin^2\delta_o} + \frac{\delta'\sin(2bk_o+\delta_o)}{4k_o^2\sin^3\delta_o} - \frac{\delta'^2\cot\delta_o}{2k_o\sin^2\delta_o} + \frac{\delta''}{4k_o\sin^2\delta_o}\right]. \quad (3-70)$$

From the last equation we can find expression for δ'';

$$\delta'' = \frac{b\cos(2bk_o+2\delta_o)}{k_o} - 2\left(\frac{\hbar^2}{2m}\right)^2 4k_o\sin^2\delta_o\int_0^b \upsilon_0(r)\upsilon_1(r)dr - \frac{\sin(2bk_o+2\delta_o)}{2k_o^2} + 2b\delta'\cot\delta_o - \frac{\delta'\sin(2bk_o+\delta_o)}{k_o\sin\delta_o} + 2\delta'^2\cot\delta_o\,. \quad (3-71)$$

To find α_2 set n = 2 into (3-44), which results in

$$\alpha_2 = \left(\frac{2m}{\hbar^2}\right)^2 \left[-\frac{\cot\delta_o}{4k_o^3} - \frac{\delta'}{4k_o^2\sin^2\delta_o} + \frac{\delta'^2\cot\delta_o}{2k_o\sin^2\delta_o} - \frac{\delta''}{4k_o\sin^2\delta_o}\right]. \quad (3-72)$$

Substituting for δ' and δ'' using (3-71) and (3-61) and using

$$2\int_0^b \upsilon_o^f(r)\upsilon_1^f(r)dr = \left(\frac{2m}{\hbar^2}\right)^2 \frac{1}{4k^3\sin^2\delta_o}(-bk_o\cos(2bk_o+2\delta_o) + \cos(bk_o+2\delta_o)\sin(bk_o) - k_o\delta'\csc\delta_o(2bk_o\cos\delta_o - 2\cos(bk_o+\delta_o)\sin(bk_o)),$$

leads to the following expressions for α_2;

$$\alpha_2 = \left(\frac{2m}{\hbar^2}\right)\left[\int_0^b 2\, \upsilon_o^f(r)\upsilon_1^f(r)\, dr - \int_0^b 2\, \upsilon_o(r)\, \upsilon_1(r) dr\right]$$
$$+ \left[\frac{b}{4k_o^2\sin^2\delta_o} + \frac{\text{b}\cos(bk_o+\delta_o)\sin bk_o}{2k_o^2\sin^3\delta_o} - \frac{b\sin(2bk_o+\delta_o)}{4k_o^2\sin^3\delta_o} - \frac{3\cot\delta_o}{4k_o^3}\right.$$
$$+ \frac{\cos(bk_o+\delta_o)}{2k_o^3\sin\delta_o} + \frac{\cos^2(bk_o+\delta_o)\sin bk_o}{k_o^3\sin^2\delta_o} - \frac{\cos(bk_o+2\delta_o)\sin bk_o}{4k_o^3\sin^2\delta_o}$$
$$- \frac{\cos(bk_o+\delta_o)\sin bk_o\sin(2bk_o+2\delta_o)}{4k_o^3\sin^3\delta_o}$$
$$- \frac{\cos(bk_o+\delta_o)\sin(2bk_o+\delta_o)}{2k_o^3\sin^2\delta_o} + \frac{\sin(2bk_o+2\delta_o)\sin(2bk_o+\delta_o)}{8k_o^3\sin^3\delta_o}$$
$$+ \frac{\cot\delta_o\sin(2bk_o+\delta_o)}{2k_o^3\sin\delta_o} - \frac{\cos(bk_o+\delta_o)\cot\delta_o\sin bk_o}{k_o^3\sin\delta_o}$$
$$\left. - \frac{\sin(2bk_o+2\delta_o)}{8k_o^3\sin^2\delta_o} + \frac{\sin(2bk_o+2\delta_o)}{8k_o^3\sin^2\delta_o}\right].$$

$$(3-73)$$

Clearly the first three terms in the second square brackets cancel each other out after using the identity sin(A+B)=sinA cosB+cosAsinB. Further the last two terms in the second brackets also cancel out. Therefore, we can write (3-73) as

$$\alpha_2 = \left(\frac{2m}{\hbar^2}\right)\left[\int_0^b 2\, \upsilon_o^f(r)\upsilon_1^f(r)\, dr - \int_0^b 2\, \upsilon_o(r)\, \upsilon_1(r) dr\right]$$
$$+ \left[-\frac{3\cot\delta_o}{4k_o^3} + \frac{\cos(bk_o+\delta_o)}{2k_o^3\sin\delta_o} + \frac{\cos^2(bk_o+\delta_o)\sin bk_o}{k_o^3\sin^2\delta_o}\right.$$
$$- \frac{\cos(bk_o+2\delta_o)\sin bk_o}{4k_o^3\sin^2\delta_o} - \frac{\cos(bk_o+\delta_o)\sin bk_o\sin(2bk_o+2\delta_o)}{4k_o^3\sin^3\delta_o}$$
$$- \frac{\cos(bk_o+\delta_o)\sin(2bk_o+\delta_o)}{2k_o^3\sin^2\delta_o} + \frac{\sin(2bk_o+2\delta_o)\sin(2bk_o+\delta_o)}{8k_o^3\sin^3\delta_o}$$
$$\left. + \frac{\cot\delta_o\sin(2bk_o+\delta_o)}{2k_o^3\sin\delta_o} - \frac{\cos(bk_o+\delta_o)\cot\delta_o\sin bk_o}{k_o^3\sin\delta_o}\right]. \qquad (3-74)$$

The last two terms in the second brackets can be simplified to get $\frac{\cot\delta_o}{2k_o^3}$. Adding this to first term results in

$$\alpha_2 = \left(\frac{2m}{\hbar^2}\right)\left[\int_0^b 2\,\upsilon_o^f(r)\upsilon_1^f(r)\,dr - \int_0^b 2\,\upsilon_o(r)\,\upsilon_1(r)dr\right] + \left[-\frac{\cot\delta_o}{4k_o^3} + \frac{\cos(bk_o+\delta_o)}{2k_o^3\sin\delta_o} + \frac{\cos^2(bk_o+\delta_o)\sin bk_o}{k_o^3\sin^2\delta_o} - \frac{\cos(bk_o+\delta_o)\sin(2bk_o+\delta_o)}{2k_o^3\sin^2\delta_o} - \frac{\cos(bk_o+2\delta_o)\sin bk_o}{4k_o^3\sin^2\delta_o} - \frac{\cos(bk_o+\delta_o)\sin bk_o\sin(2bk_o+2\delta_o)}{4k_o^3\sin^3\delta_o} + \frac{\sin(2bk_o+2\delta_o)\sin(2bk_o+\delta_o)}{8k_o^3\sin^3\delta_o}\right]. \quad (3-75)$$

Similarly, the last three terms in the second brackets simplify to $\frac{\cot\delta_o}{4k_o^3}$ which cancel out with the first term leading to

$$\alpha_2 = \left(\frac{2m}{\hbar^2}\right)\left[\int_0^b 2\,\upsilon_o^f(r)\upsilon_1^f(r)\,dr - \int_0^b 2\,\upsilon_o(r)\,\upsilon_1(r)dr\right] + \left[\frac{\cos(bk_o+\delta_o)}{2k_o^3\sin\delta_o} + \frac{\cos^2(bk_o+\delta_o)\sin bk_o}{k_o^3\sin^2\delta_o} - \frac{\cos(bk_o+\delta_o)\sin(2bk_o+\delta_o)}{2k_o^3\sin^2\delta_o}\right]. \quad (3-76)$$

The last equation can be written as

$$\alpha_2 = \left(\frac{2m}{\hbar^2}\right)\left[\int_0^b 2\,\upsilon_o^f(r)\upsilon_1^f(r)\,dr - \int_0^b 2\,\upsilon_o(r)\,\upsilon_1(r)dr\right] + \left[\frac{\cos(bk_o+\delta_o)\sin\delta_o}{2k_o^3\sin^2\delta_o} + \frac{2\cos^2(bk_o+\delta_o)\sin bk_o}{2k_o^3\sin^2\delta_o} - \frac{\cos(bk_o+\delta_o)\sin(2bk_o+\delta_o)}{2k_o^3\sin^2\delta_o}\right]. \quad (3-77)$$

Using the identity cos(A+B)=cosA cosB-sinA sinB , we can rewrite the last equation as

$$\alpha_2 = \left(\frac{2m}{\hbar^2}\right)\left[\int_0^b 2\,\upsilon_o^f(r)\upsilon_1^f(r)\,dr - \int_0^b 2\,\upsilon_o(r)\,\upsilon_1(r)dr\right] + \left[\frac{\cos(bk_o+\delta_o)}{2k_o^3\sin^2\delta_o}(\sin\delta_o + 2\cos bk_o\cos\delta_o\sin bk_o - 2\sin bk_o\sin bk_o\sin\delta_o - \sin 2bk_o\cos\delta_o - \cos 2bk_o\sin\delta_o)\right], \quad (3-78)$$

Clearly, the second and the fourth terms in the second bracket cancel out, leading to

$$\alpha_2 = \left(\frac{2m}{\hbar^2}\right)\left[\int_0^b 2\,\upsilon_o^f(r)\upsilon_1^f(r)\,dr - \int_0^b 2\,\upsilon_o(r)\,\upsilon_1(r)dr\right] + \left[\frac{\cos(bk_o+\delta_o)}{2k_o^3\sin^2\delta_o}(\sin\delta_o - 2\sin^2 bk_o\sin\delta_o - \cos 2bk_o\sin\delta_o)\right], \quad (3-79)$$

The last there terms cancel each other after using the identity $\cos 2x = 1 - 2\sin^2 x$. Our final result for α_2 now reads

$$\alpha_2 = \left(\frac{2m}{\hbar^2}\right)\left[\int_0^b 2\,\upsilon_o^f(r)\upsilon_1^f(r)\,dr - \int_0^b 2\,\upsilon_o(r)\,\upsilon_1(r)dr\right]. \quad (3-80)$$

An alternative and simpler way to obtain α_2 (and higher orders) starts by noting that (3-11) and (3-43) are valid for all values of k_0. Hence, we may write

$$\upsilon_{n+1}(k_o,r) = \frac{2m}{\hbar^2}\frac{\partial}{\partial k^2}\upsilon_n(k,r)\Big|_{k\,=\,k_o}, \quad (3-81)$$

and

$$\alpha_{n+1} = \left(\frac{2m}{\hbar^2}\frac{\partial}{\partial k^2}\right)^{n+1} k\cot\delta\,\Big|_{k\,=\,k_o} = \left(\frac{2m}{\hbar^2}\right)\frac{\partial}{\partial k^2}\alpha_n\,\Big|_{k\,=\,k_o}, \quad (3-82)$$

where we have used (3-44) to obtain the last expression. Further, defining a power series expansion in ΔE for $\upsilon^f(k,r)$ similar to that given in (3-11), then $\upsilon_{n+1}^f(k_o,r)$ may be expressed as

$$\upsilon_{n+1}^f(k_o,r) = \frac{2m}{\hbar^2}\frac{\partial}{\partial k^2}\upsilon_n^f(k,r)\Big|_{k\,=\,k_o}. \quad (3-83)$$

Consequently, using the last three equations in addition to (3-64) we obtain

$$\alpha_2 = \left(\frac{2m}{\hbar^2}\right)\left[\int_0^b 2\,\upsilon_o^f(r)\upsilon_1^f(r)\,dr - \int_0^b 2\,\upsilon_o(r)\,\upsilon_1(r)dr\right]. \quad (3-84)$$

Which is the same as (3-80).

Chapter Four

Data and Results

Data and Results

In this chapter we shall test the validity of the derived expressions for the changes in the phase shifts in the s-wave case when a small perturbation in the incident energy is introduced. This will be done by considering three examples. The first will assume both the local and velocity-dependent parts of the potential to have the form of a finite square well of finite common radius b. The second assumes the local potential to be a square well, but the velocity-dependent part is taken to have a Saxon-Woods form. The third one assumes both the local and the velocity dependent parts of the potential to have the Saxon-Woods form. In the first example we shall solve Schrödinger's equation exactly and determine the exact changes in the phase shifts due to a small change in the energy of the incident particle. In the second and third examples we cannot find exact analytical solutions to Schrödinger equation. Consequently, we shall determine the exact phase shift changes numerically. Then we shall use the perturbation formulas derived in Chapter Three to calculate the changes in the phase shifts. The validity of the derived perturbation formalism shall be tested by comparing the perturbed phase shifts to the corresponding exactly obtained ones.

4-1 Finite square-well potential

In this example we consider both the local and velocity-dependent parts of the potential to have the form of a finite square-well; this potential can be written as

$$V(r,p) = -V_0\theta(r-b) + \frac{\hbar^2}{M}\nabla.\rho(r)\nabla, \qquad (4-1)$$

where,

$$\theta(r-b) = 1, \quad r < b, \qquad \theta(r-b) = 0, \quad r > b, \qquad (4-2)$$

where $\theta(r-b)$ is the step function.

and

$$\rho(r) = \beta\theta(r-b), \qquad (4-3)$$

where β is a constant and M is the nucleon mass.

The corresponding s-wave time-independent Schrödinger equation is exactly solvable and has the form

$$[1-\beta\theta(r-b)]\upsilon''(r)+\left[k^2+\frac{MV_0}{\hbar^2}\theta(r-b)\right]\upsilon(r) \\ =-\beta\left[\upsilon'(r)-\frac{\upsilon(r)}{r}\right]\delta(r-b), \qquad (4-4)$$

where,

$$k^2=\frac{M(E_o+\Delta E)}{\hbar^2}. \qquad (4-5)$$

and $E=E_o+\Delta E$ is the energy available in the center of mass of the two interacting nucleons. The delta function on the right-hand side arises due to the sharp edge at r = b. In terms of the radial wave function $R_\ell(k,r)=v_\ell/$r we can write (4-4) as

$$[1-\beta\theta(r-b)]R_o''(k,r)+\frac{2}{r}[1-\beta\theta(r-b)]R_o'(k,r)+\left[k^2+\frac{MV_0}{\hbar^2}\theta(r-b)\right]R_o(k,r) \\ =-\beta\delta(r-b)R_o'(k,r). \qquad (4-6)$$

For r < b, the radial equation is

$$(1-\beta)R_o''(K,r)+(1-\beta)\frac{2}{r}R_o'(K,r)+\frac{M}{\hbar^2}(V_0+E_o+\Delta E)R_o(K,r) \\ =0. \qquad (4-7)$$

Or,

$$R_o''(K,r)+\frac{2}{r}R_o'(K,r)=-\frac{M}{\hbar^2}\frac{(V_0+E_o+\Delta E)}{(1-\beta)}R_o(K,r). \qquad (4-8)$$

The general solution is given by (2-49); namely

$$R_o^I(K,r)=N_o(K)j_o(Kr), \qquad r<b \qquad (4-9)$$

where $K^2=\frac{M}{\hbar^2}\left(\frac{V_0+E_0+\Delta E}{1-\beta}\right)$.

For r > b, the radial equation is

$$R_o''(\mathrm{k},r)+\frac{2}{r}R_o'(\mathrm{k},r)+\frac{M}{\hbar^2}(E_o+\Delta E)R_o(\mathrm{k},r)=0. \qquad (4-10)$$

And, according to (2-51) the general solution is,

$$R_o^e(k,r) = B_o(k)[j_o(kr) - \tan\delta_o(k)n_o(kr)], \qquad (4-11)$$

where $k^2 = \frac{M}{\hbar^2}(E_o + \Delta E)$.

The solutions for r < b and r > b can be joined smoothly at r = b by requiring that

$$R_o^I(K,b) = R_o^e(k,b). \qquad (4-12)$$

Evaluating for the explicit expressions results in

$$N_o(K)j_o(Kb) = B_o(k)[j_o(kb) - \tan\delta_o(k)n_o(kb)]. \qquad (4-13)$$

It is worth noting that there is a finite jump in the derivative of the wave function at the boundary ($r = b$) due to the sharp edge of the Kisslinger potential. The behavior of the derivative of the wave function at the boundary can be determined by integrating Schrödinger's equation across the boundary from b-ε to b+ε and then taking the limit as $\varepsilon \longrightarrow 0$. We shall drop the subscript (o) for clarity of presentation and it is understood that we are dealing with the s-wave case.

To accomplish this, we start with equation (3-7) which can be written as

$$\frac{d}{dr}\left[r^2(1-\rho(r))\frac{dR(k,r)}{dr}\right] - \frac{2m}{\hbar^2}[V(r)-E]rR(k,r) = 0. \qquad (4-14)$$

Integrating the above equation from b-ε to b+ε results in

$$\int_{b-\varepsilon}^{b+\varepsilon}\frac{d}{dr}\left[r^2(1-\rho(r))\frac{dR(k,r)}{dr}\right]dr - \int_{b-\varepsilon}^{b+\varepsilon}\frac{2m}{\hbar^2}[V(r)-E]rR(k,r)dr = 0. \qquad (4-15)$$

In the limit $\varepsilon \longrightarrow 0$ the second integral vanishes as it becomes an integral over a point. Therefore, we can write the last equation as

$$\left[r^2(1-\rho(r))\frac{dR(k,r)}{dr}\right]_{b-\varepsilon}^{b+\varepsilon} = 0. \qquad (4-16)$$

Evaluating the integration limits gives

$$(b+\varepsilon)^2\left(1-\rho(b+\varepsilon)\right)\frac{dR(k,b+\varepsilon)}{dr}-(b-\varepsilon)^2\left(1-\rho(b-\varepsilon)\right)\frac{dR(K,b-\varepsilon)}{dr} = 0. \qquad (4-17)$$

Taking the limit as $\varepsilon \longrightarrow 0$, we obtain

$$b^2\frac{dR(k,b)_>}{dr}-b^2\left(1-\rho(b)\right)\frac{dR(K,b)_<}{dr}=0, \qquad (4-18)$$

where we used the fact $\rho(b+\varepsilon)=0$. Finally we get

$$\frac{dR(k,b)_>}{dr}=(1-\beta)\frac{dR(K,b)_<}{dr}, \qquad (4-19)$$

where $\frac{dR(k,b)_>}{dr}$ and $\frac{dR(K,b)_<}{dr}$ designate the derivatives of the radial wave function with respect to r, in the internal and external regions, respectively.

Using equations (4-12) , (4-19), and following the same procedure as in Section (2-3) an exact expression for tanδ is

$$\tan\delta_0(k)=\frac{k\,j_0(Kb)j_0'(kb)-(1-\beta)Kj_0'(Kb)j_0(kb)}{k\,n_0'(kb)j_0(Kb)-(1-\beta)\,K\,j_0'(Kb)\,n_0(kb)}. \qquad (4-20)$$

The above equation allows an exact determination of the phase shift. Note that the above expression for the phase shift reduces to that given in equation (2 - 55) in the limit $\beta \to 0$.

Our forms V_0 and ρ are those adopted by Razavy et al. who modeled the neutron-proton scattering at low energy using a velocity-dependent potential. They managed to reproduce the ^{1}S, ^{1}D and ^{1}G singlet-even phase shifts for the proton-neutron interaction. Therefore, we shall adopt their values for V_0 , β and b, namely β= - 0.21, V_0=16.9MeV, b=2.4 fm. Substituting the above values in (4-20) we find the exact values of the perturbed phase shifts as a function of the energy E which is perturbed by a small amount ΔE=0.1 E_0 from the original energy E_0. The exact perturbed phase shifts are listed in Table (4-1). Similarly, by switching off the perturbation (i.e. setting $\Delta E=0$)

we used the same equation to calculate the exact unperturbed phase shift values δ_0, which are shown in the second column of Table (4-1).

To find the values of the phase shift using the perturbation formulas, let's start from the expansion of $k\ \cot\delta$; namely

$$k\ \cot\delta = \alpha_o + \frac{\hbar^2}{2m}(k^2 - k_o^2)\alpha_1 + \left(\frac{\hbar^2}{2m}\right)^2 \frac{(k^2 - k_o^2)^2\alpha_2}{2!} + \cdots \qquad (4-21)$$

If we substitute $k^2 = k_0^2$ then,

$$k_o\ \cot\delta_o = \alpha_o\ , \qquad (4-22)$$

where $k_o{}^2 = \frac{ME_0}{\hbar^2}$.

From (4-22) we find the values of α_o which are shown in the fourth column of Table (4-1).

To find the values of α_1 we start from (3-64), namely

$$\alpha_1 = \frac{2m}{\hbar^2}\left[\int_0^b (\upsilon_o^f(r))^2 dr - \int_0^b \upsilon_o^2(r) dr\right], \qquad (4-23)$$

where, the unperturbed wave functions $\upsilon_o^f(r)$ and $\upsilon_o(r)$ are given by

$$\upsilon_o^f(r) = \frac{\sin\,(k_o r + \delta_o)}{\sin\,\delta_o}. \qquad (4-24)$$

and

$$\upsilon_0(r) = \frac{A\ \sin K_0 r}{K_0}, \qquad r < b \qquad \text{and} \qquad K_0^2 = \frac{M}{\hbar^2}\left(\frac{V_0 + E_0}{1-\beta}\right). \qquad (4-25)$$

Note that $\upsilon_o^f(r)$ is the free particle wave function which is defined for all r and has the same form as $\upsilon_o(r)$ in the asymptotic region; while $\upsilon_o(r) = A \sin K_0 r\ /K_0$ is the unperturbed wave function of the scattered particle in the internal region. The value of A is determined using the boundary condition $\upsilon_o(b) = \upsilon_o^f(b)$, i.e,

$$A = \frac{K_0 \sin(k_o b + \delta_o)}{\sin\delta_o\ \sin K_0 b}. \qquad (4-26)$$

Substituting $\upsilon_o^f(r)$ and $\upsilon_o(r)$ into (4-23) and evaluating the integral, results in the value of α_1. The values of α_1 for different values of energies are shown in column 5 of Table (4-1).

To determine α_2 we start from (3-80);

$$\alpha_2 = \left(\frac{2m}{\hbar^2}\right)\left[\int_0^b 2\,\upsilon_o^f(r)\upsilon_1^f(r)\,dr - \int_0^b 2\,\upsilon_o(r)\,\upsilon_1(r)dr\right], \qquad (4-27)$$

where $\upsilon_1^f(r)$ is the first order correction to the wave function in the asymptotic region which is given by equation (3-14), namely

$$\upsilon_1^f(r) = \frac{2m}{\hbar^2}\left(\frac{r\cos(k_o r + \delta_o)}{2k_o \sin\delta_o} - \frac{\delta' \sin k_o r}{2k_o \sin^2\delta_o}\right) \qquad (4-28)$$

where δ' is given by (3-61). Here $\upsilon_1(r)$ is the first-order wave function correction in the internal region which is determined by solving (3-18) numerically. It is worth noting that the point r = 0 is a regular singular point of the differential equation (3-18) which means that the solution to the differential equation is finite at r = 0 (see Appendix A). The values of α_2 for different energies are shown in column 6 of Table (4-1). We used the values of α_o, α_1 and α_2 to determine the perturbed phase shifts δ^p up to second order using (4-21) corresponding to an energy change $\Delta E = 0.1E_0$.

$E_{0\ CM}$ (MeV)	$\delta_0(\Delta E=0)$	$\delta(\Delta E=0.1\ E_0)$ exact phase shift	α_0	α_1	$\alpha_2(10^{-3})$	δ^p (perturbed phase shift)
5	1.015	0.999	0.21560	0.03690	-0.133	1.000
10	0.870	0.844	0.41403	0.04274	-0.025	0.846
15	0.750	0.717	0.64561	0.05021	-0.017	0.720
50	0.217	0.166	4.98145	0.30324	0.558	0.212
90	-0.083	-0.121	-17.70860	0.91221	-22.791	-0.058

Table(4-1): The third column gives the exact perturbed phase shift values δ when the incident energy is perturbed by a small amount $\Delta E=0.1E_0$. Columns 4-6 give the parameters of the effective range expansion as in Equation (4-21). The perturbed phase shifts calculated up to second order are shown in the last column.

Clearly, the agreement between the perturbed phase shifts (δ^p) and the corresponding exact ones (δ) shown in the third and seventh columns of Table 4-1, respectively, is quite good at low energies where $\Delta E=0.1\ E_0$ is small. This can be understood by noting that $k\ \cot\delta$ in (4-21) is expanded in powers of ΔE. As ΔE gets larger the higher order terms in the effective range expansion become significant and the accuracy of the results of the derived formalisms become progressively poorer. This is clearly seen by inspecting the values of δ and δ^p corresponding to E_0 =50 and 90 MeV as shown in table (4-1).

Using the same potential forms we determined the perturbed scattering phase shifts corresponding to a constant energy perturbation $\Delta E=0.5$ MeV from the original energy E_0. As shown in Table (4-2) the agreement between the exact perturbed phase shifts δ and the corresponding perturbed phase shifts δ^p determined up to second order is quite good for all energies.

$E_{0\ CM}$ (MeV)	δ_0 (ΔE=0)	δ(ΔE=0.5MeV) exact phase shift	α_0	α_1	α_2 (10^{-3})	δ^p (perturbed phase shift)
5	1.015	0.999	0.21560	0.03690	-3.930	1.000
10	0.870	0.857	0.41403	0.04274	-1.364	0.857
15	0.750	0.739	0.64561	0.05021	-0.0051	0.739
50	0.217	0.212	4.98145	0.30324	55.921	0.212
90	-0.083	-0.085	-17.70860	0.91221	-68.784	-0.085

Table(4-2): The third column gives the exact perturbed phase shift values δ when the incident energy is perturbed by a small amount ΔE=0.5MeV. Columns 4-6 give the parameters of the effective range expansion as in equation (4-21). The perturbed phase shifts calculated up to second order are shown in the last column.

4-2 Finite square-well local potential and Saxon-Woods form for ρ(r)

In this example we consider the local potential to have the form of a finite square well, but the velocity-dependent part is taken to have the form of a Saxon-Woods potential; which can be written as

$$\rho(r) = \frac{-\rho_0}{1 + e^{\left(\frac{r-R}{d}\right)}}, \qquad (4-29)$$

where ρ_0 represents the potential depth, d is a length which is a measure of the surface thickness of the nucleus, and R is the mean radius. The parameters ρ_0, d, and R were chosen such that we recovered, as closely as possible, the exact unperturbed scattering phase shifts δ_0 given in Table (4-1). The values are

ρ_0 =0.6, R = 1.7fm d = 0.1 fm $(4-30)$

While the values of the square well parameters were chosen as in the previous example.

Figure (4-1) represents the Saxon-Woods potential corresponding to the above values. The Schrödinger equation subjected to Saxon-Woods potential cannot be solved analytically and must be treated numerically.

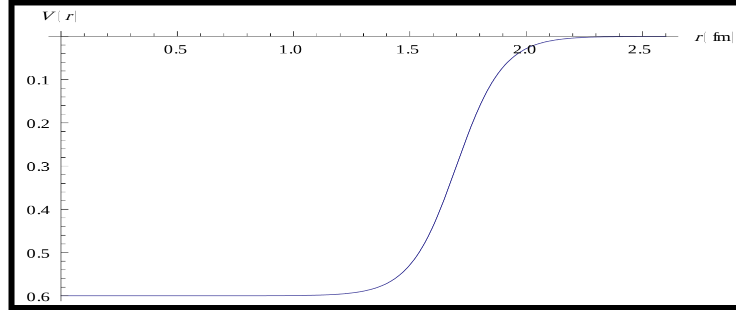


Figure (4-1) Saxon-Woods potential.

To find the unperturbed phase shift δ_0 we solved Schrodinger's equation (3-7) numerically to find the unperturbed wave functions in the internal region r < b and the asymptotic region r > b. Since $\rho(r)$ is continuous, the wave function and its derivative are continuous at r = b.

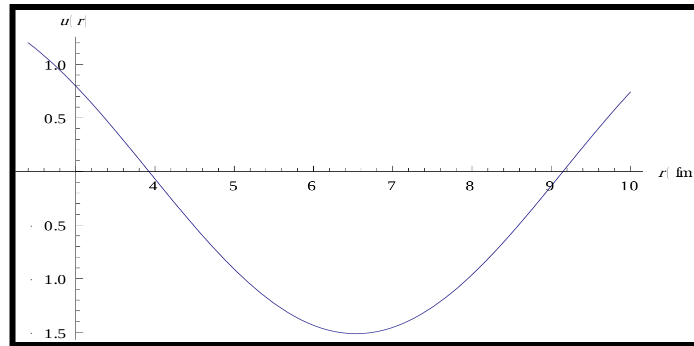


Figure (4-2): The wave function versus r corresponding to an incident energy of 15 MeV.

By determining the interception of the external wave function with *r-axis* (for example, for E= 15 MeV the first interception point occurs at r_0 =3.9232fm). The exact unperturbed phase shift is then calculated by setting

$$\sin(k\, r_0 + \delta) = \sin kr_0 \cos\delta + \cos kr_0 \sin\delta = 0, \qquad (4-31)$$

we determined larger values of r where $\upsilon(r)$ intercepts the r-axes and the corresponding values of δ were the same. Similarly, the exact unperturbed phase shift values were determined as a function of the incident energy and listed in the second column of Table (4-3). To find the values of the effective range α_1 we used (4-23) where $\upsilon_o^f(r)$ is given by (4-24) but due to the form of $\rho(r)$ no analytic expression for $\upsilon_o(r)$ in the internal region could be obtained, hence we had to determine it numerically. The numerical values for α_1 corresponding to $\Delta E=0.1E_0$ are listed in column 5 of Table (4-3).

Similarly, we solved (3-18) numerically to determine $\upsilon_1(r)$ in the internal and external regions and then used (4-27) to evaluate the shape parameter α_2 whose values are given in column 6 of Table

(4-3) corresponding to $\Delta E=0.1E_0$. Substituting α_0 , α_1 , and α_2 into (4-21), the perturbed phase shifts were determined and listed in column 7 of Table (4-3).

$E_{0\ CM}$ (Mev)	δ_0 ($\Delta E=0$)	$\delta(\Delta E=0.1\ E_0)$ exact phase shift	α_0	α_1	¶ $(10^{-3})\alpha_2$	δ^p (perturbed phase shift)
5	1.045	1.029	0.20149	0.03513	0.156	1.029
10	0.901	0.875	0.38911	0.04004	0.892	0.875
15	0.783	0.751	0.60418	0.04616	1.862	0.751
50	0.261	0.209	4.10361	0.22006	92.491	0.206
90	-0.086	-0.146	17.0825	1.30051	-678.280	-0.152

Table(4-3): The third column gives the exact perturbed phase shift values δ when the incident energy is perturbed by a small amount $\Delta E=0.1E_0$. Columns 4-6 give the parameters of the effective range expansion as in Equation (4-21). The perturbed phase shifts calculated up to second order are shown in the last column.

Clearly, the agreement between the perturbed phase shift (δ^p) and the corresponding exact ones (δ) shown in the third and seventh columns of Table 4-3, respectively, is quite good at low energies when $\Delta E=0.1\ E_0$ is small. As before, this can be understood by noting that $k\cot\delta$ in (4-21) is expanded in powers of ΔE. As ΔE gets larger the higher order terms in the effective range expansion become significant and the accuracy of the results of the derived formalisms become progressively poorer.

Using the same potential forms we determined the perturbed scattering phase shifts corresponding to a constant energy perturbation $\Delta E=0.5$ MeV away from the original energy E_0. Since $\Delta E < 1$ MeV, the agreement between the exact perturbed phase shifts δ and the perturbed phase shifts δ^p determined up to second order is quite good for all energies as can be verified by inspecting Table (4-4).

$E_{0\ CM}$ (Mev)	δ_0 ($\Delta E=0$)	$\delta(\Delta E=0.5\text{MeV})$ exact phase shift	α_0	α_1	α_2 (10^{-3})	δ^p (perturbed phase shift)
5	1.045	1.029	0.20149	0.03513	0.156	1.029
10	0.901	0.888	0.38911	0.04004	0.892	0.888
15	0.783	0.772	0.60418	0.04616	1.862	0.772
50	0.261	0.256	4.10361	0.22006	92.491	0.256
90	-0.086	-0.089	-17.0825	1.30051	-678.280	-0.088

Table(4-4): The third column gives the exact perturbed phase shift values δ when the incident energy is perturbed by a small amount $\Delta E=0.5$MeV. The perturbed phase shifts calculated up to second order are shown in the last column.

4-3 Saxon-Woods local potential and Saxon-Woods form for ρ(r)

In this example we consider both the local and velocity-dependent parts of the potential to have the form of a Saxon-Woods potential. To find the unperturbed phase shift δ_0 we solved the Schrödinger equation (3-7) numerically, and determined the

unperturbed wave functions in the internal (r < b) and asymptotic (r > b) regions. Since ρ(r) is continuous, the wave function derivatives are continuous at r = b.

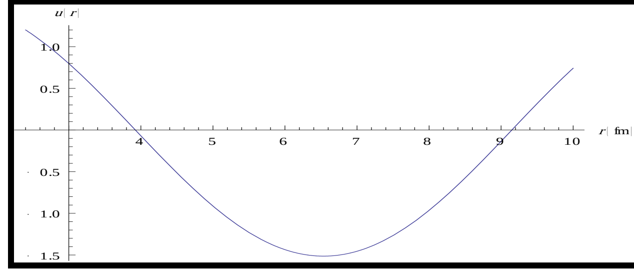


Figure (4-3). The wave function versus r corresponding to an incident energy of 15 MeV.

The phase shifts and the effective range expansion parameters were determined in the same way as in the previous two examples and the values of δ_0 are listed in the second column of Table (4-5). The values of α_1 corresponding to $\Delta E=0.1\ E_0$ are shown in column 5 of Table (4-5), while the value of α_2 are listed in column 6. We used the values of α_0, α_1, and α_2 and hence determined the perturbed phase shifts δ^p up to second order using Equation (4-21) corresponding to an energy change $\Delta E=0.1\ E_0$.

$E_{0\ CM}$ (Mev)	$\delta_0(\Delta E=0)$	$\delta(\Delta E=0.1\ E_0)$ exact phase shift	α_0	α_1	$\alpha_2(10^{-3})$	δ^p (perturbed phase shift)
5	-0.054	-0.057	-6.42159	-0.06201	-9.958	-0.056
10	-0.079	-0.085	-6.13255	-0.00949	10.298	-0.084
15	-0.105	-0.112	-5.72067	0.02364	20.249	-0.110
50	-0.325	-0.362	-3.25676	0.03785	18.125	-0.359
90	-0.636	-0.708	-1.99469	0.02043	18.866	-0.733

Table(4-5): The third column gives the exact perturbed phase shift values δ when the incident energy is perturbed by a small amount $\Delta E=0.1E_0$. Columns 4-6 give the parameters of the effective range expansion as in Equation (4-21). The perturbed phase shifts calculated up to second order are shown in the last column.

The exact phase shift is obtained by solving Schrödinger equation with ρ_0=8 MeV, b = 2.4, R=1.3 fm, and a=0.2 fm. We chose these values such that ρ(r) and the local potential vanish at the same value of the radial distance (i.e they have a common radius beyond which they vanish). It is worth mentioning that it was not possible to define a set of potential parameters that could reproduce the experimentally determined phase shifts. From Table (4-5) above, for E = 5MeV the exact phase shift δ = -0.057 in a good agreement with -0.056 obtained using the perturbation approach. At small energy the agreement is expected to be good as we mentioned earlier. However, as expected, as the energy perturbation becomes larger, the discrepancy between the exact and perturbed phase shifts grows progressively larger as is evident in Table (4-5).

Chapter Five
Discussion and Conclusions

Discussion and Conclusions

Our aim has been to study the energy dependence of the scattering phase shifts when the incident energy is changed by a small amount ΔE from its unperturbed value E_0. This is done by considering the time-independent Schrödinger equation where the potential is assumed to be velocity-dependent. We expanded $k \cot\delta$, where δ is the perturbed phase shift, as a power series in ΔE and derived analytical formulas for the effective range expansion parameters to second order in the perturbing energy. Also we obtained a general formula for the wave function correction when a small perturbation to the incident energy is present. At low energies, the Bethe formula for the effective range has been reproduced. In deriving our formalism we have considered a velocity-dependent potential which is relevant to fields like nuclear physics (Kisslinger, 1955) and atomic physics (Green and Ueda, 1981). In addition, determining the variation of the scattering phase shifts with energy is relevant to the time delay in the emergence of the scattered particles (Romo and Valluri, 1998). Although we have restricted our formalism to the simpler s-wave case. It is possible to generalize the derived formulas to higher order waves.

In order to test the validity of the derived formalism, we have considered three examples. The first assumed the local and velocity-dependent parts of the potential to have the form of a finite square well of finite radius b. Due to the simple forms of V(r) and ρ(r) we obtained analytical expressions for the exact perturbed phase shifts δ which were used to determine the phase shifts as a function of the incident energy. The values of the phase shifts are presented in the third column of Table (4-1) corresponding to a variable perturbing energy $\Delta E = 0.1E_0$ Using the developed perturbation formulas given in (4-22), (4-23) and (4-27) we calculated α_0, α_1 and α_2, respectively, which are presented in column 4-6 of the same table. Further, the perturbed phase shifts δ^p

calculated up to second order using (4-21) are shown in the last column of Table (4-1). Clearly, when the perturbing energy is small, the agreement between δ and δ^p is quite good. We have also determined the perturbed phase shifts corresponding to a constant perturbing energy ΔE = 0.5MeV, as shown in Table (4-2). The derived perturbation formalism has resulted in very accurate perturbed phase shifts regardless of the value of the initial unperturbed energy. The accuracy of the perturbed phase shift values depends on the size of ΔE. As discussed in the last chapter, when $\Delta E < 1$, we expect the derived formalism to produce accurate results. As the energy perturbation gets larger, the accuracy of the perturbed results becomes progressively poorer.

The second example assumed the local potential to be a square well, but the velocity-dependent part is taken to have a Saxon-Woods potential form. The third one assumed both the local and the velocity dependent parts of the potential to have Saxon-Woods forms. For such potentials we cannot find an analytical exact solution to Schrödinger equation. Consequently, we determined the exact phase shift changes numerically. Then we used the developed perturbation formulas given in (4-22), (4-23) and (4-27) to calculate α_0, α_1 and α_2, respectively, which are presented in columns 4-6 of Table (4-3), (4-4) and (4-5). The perturbed phase shifts δ^p calculated up to second order using (4-21) are shown in the last column of these Tables. The agreement between the perturbed phase shifts δ^p and the corresponding ones δ is quite good at low energies where $\Delta E=0.1E_0$ is small. Since $k \cot\delta$ is expanded in powers of ΔE, then as ΔE gets larger the higher order terms in the effective range expansion become more significant and the accuracy of the results of the derived formalism gets progressively poorer. For constant perturbing energy ΔE=0.5 MeV the derived formalism resulted in accurate perturbed results for all energies as the energy perturbation is small.

The derived formalism is relevant to the field of nuclear physics where the p-wave nature of the pion-nucleon scattering has been correctly described using a velocity-dependent potential (Kisslinger, 1955). Further, such a potential has also been used in modeling the scattering of electrons from atomic neon and oxygen (Green and Ueda, T., 1981). In a more recent work, it has been shown that the time delay in the emergence of the scattered particles depends essentially on the variation of the scattering phase shifts with the incident energy (Romo and Valluri, 1998).

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wikipedia.org/wiki/Woods%E2%80%93Saxon_potential

Appendix A

Regular singular points

Start from equation (3-7) and rewrite this equation as

$$\upsilon''(k,\mathrm{r}) - \frac{\rho'(r)}{\left(1-\rho(r)\right)}\upsilon'(k,\mathrm{r}) + \frac{1}{\left(1-\rho(r)\right)}\left[k^2 + \frac{\rho'(r)}{r} - U(r)\right]\upsilon(k,r) = 0, \qquad (A-1)$$

which in the standard form, is expressed as

$$\upsilon''(k,\mathrm{r}) + P(r)\upsilon'(k,\mathrm{r}) + Q(r)\upsilon(k,r) = 0. \qquad (A-2)$$

Equation (A-2) shows that if the functions $P(r)$ and $Q(r)$ remain finite at $r = r_0$, point $r = r_0$ is an ordinary point. In contrast if either $P(r)$ or $Q(r)$ (or both) diverges as $r \longrightarrow r_0$ then point r_0 is a singular point. Moreover, if $(r - r_0)P(r)$ and $(r - r_0)^2Q(r)$ remain finite as $r \longrightarrow r_0$, then $r = r_0$ is called a regular singular point.

Back to equation (A-1) and comparing it with (A-2) we have

$$P(r) = \frac{-\rho'(r)}{\left(1-\rho(r)\right)}, \qquad (A-3)$$

$$Q(r) = \frac{1}{\left(1-\rho(r)\right)}\left[k^2 + \frac{\rho'(r)}{r} - U(r)\right]. \qquad (A-4)$$

Which shows that $\frac{\rho'(r)}{(1-\rho(r))r}$ diverges as $r \longrightarrow 0$, hence $r = 0$ is a singular point. But $r^2\frac{\rho'(r)}{(1-\rho(r))r}$ remains finite as $r \longrightarrow 0$, then $r = 0$ is a regular singular point.

Equation (A-1) is homogeneous because each term contains $\upsilon(k,r)$ or a derivative, we shall attempt to solve this linear, second-order differential equation by substituting in a power series with undetermined coefficients as (Arfken, 2005)

$$\upsilon(k,r) = \sum_{n=0}^{\infty} a_n r^{n+s}, \qquad (A-5)$$

Where $a_0 \neq 0$ by definition. By differentiating twice, we obtain

$$v'(k,r) = \sum_{n=0}^{\infty} a_n (n+s) r^{n+s-1}, \qquad (A-6)$$

$$v''(k,r) = \sum_{n=0}^{\infty} a_n (n+s)(n+s-1) r^{n+s-2}, \qquad (A-7)$$

by substituting into (A-1) we have

$$\sum_{n=0}^{\infty} a_n (n+s)(n+s-1) r^{n+s-2} - \frac{\rho'(r)}{(1-\rho(r))} \sum_{n=0}^{\infty} a_n (n+s) r^{n+s-1} + \frac{1}{(1-\rho(r))}\left[k^2 + \frac{\rho'(r)}{r} - U(r)\right] \sum_{n=0}^{\infty} a_n r^{n+s} = 0. \quad (A-8)$$

To simplify further, equation (A-8) can be represented as

$$\sum_{n=0}^{\infty} a_n (n+s)(n+s-1) r^{n+s-2} + \frac{\rho'(r)}{(1-\rho(r))} \sum_{n=0}^{\infty} a_n (1-n-s) r^{n+s-1} + \frac{1}{(1-\rho(r))}[k^2 - U(r)] \sum_{n=0}^{\infty} a_n r^{n+s} = 0. \qquad (A-9)$$

The lowest power of r is r^{s-2} corresponding to n = 0 in the first summation. Therefore the quantity s must satisfy the indicial equation

$$a_0 s(s-1) = 0, \qquad (A-10)$$

we had chosen a_0 as the coefficient of the lowest nonvanishing terms of the series (A-5), hence, by definition $a_0 \neq 0$. Therefore we have

$$s(s-1) = 0, \qquad (A-11)$$

this leads to $s = 0$ or $s = 1$. The choice $s = 0$ leads to a solutions which does not satisfy the condition $v_n(0) = 0$. However, the other choice $s = 1$ results in a solution that satisfies the requirement that $v_n(0)$ must vanish at the origin.

In order to find the recursion relation relating the differential coefficient we set $n = j + 1$ in the first summation, $n = j$ in the second and $n = j - 1$ in the third summation. This results in

$$\sum_{n=0}^{\infty} a_{j+1}(j+2)(j+1)r^j - \frac{\rho'(r)}{(1-\rho(r))}\sum_{n=0}^{\infty} a_j(j)r^j + \frac{1}{(1-\rho(r))}[k^2 - U(r)]\sum_{n=0}^{\infty} a_{j-1}r^j = 0. \qquad (A-12)$$

We can write the last equation as

$$\sum_{n=0}^{\infty} a_{j+1}(j+2)(j+1)r^j - \frac{\rho'(r)}{(1-\rho(r))}\sum_{n=0}^{\infty} a_j(j)r^j + \frac{1}{(1-\rho(r))}[k^2 - U(r)]\sum_{n=0}^{\infty} a_{j-1}r^j = 0. \qquad (A-13)$$

Or,

$$a_{j+1} = \frac{a_j\rho'(r)j - [k^2 - U(r)]a_{j-1}}{(1-\rho(r))(j+2)(j+1)} \qquad (A-14)$$

This is three terms recursion relation. Given a_j, we may calculate a_{j-1}, a_{j+1} and then a_{j+1} and so on as is desired.

Equation (A-14) leads to

$a_{1=}0,$

$$a_2 = \frac{-[k^2 - U(r)]a_0}{6\,(1-\rho(r))}$$

Consequently, our solution in the vicinity of the origin is

$$v(k,r) = [a_0 r + a_2 r^3 + \cdots], \qquad (A-15)$$

Where a_0 is an overall constant which is taken to be unity. This choice does not affect the zeros of the wave function in the asymptotic region and hence has no effect on the value of the phase shifts.

Appendix B
Determination of exact phase shift for square well Saxon Woods potential

```
(*energy available in the CM of the two interacting nucleon*)En = 5;
(*depth of the local potential*)V0 = 16.9;
(*nucleon mass*)M = 938.0;
(*range of the potential*)b = 2.4;
(*Planck constant*)hc = 197.3;
```

$$k = \sqrt{\frac{M\,En}{(hc)\,^\wedge 2}};$$

$$\kappa = \sqrt{\frac{M\,(En + V0)}{(hc)\,^\wedge 2}};$$

```
(*parametrs of Saxon-Woods potential*)
ρ0 = 0.6;
R = 1.7;
d = 0.1;
```

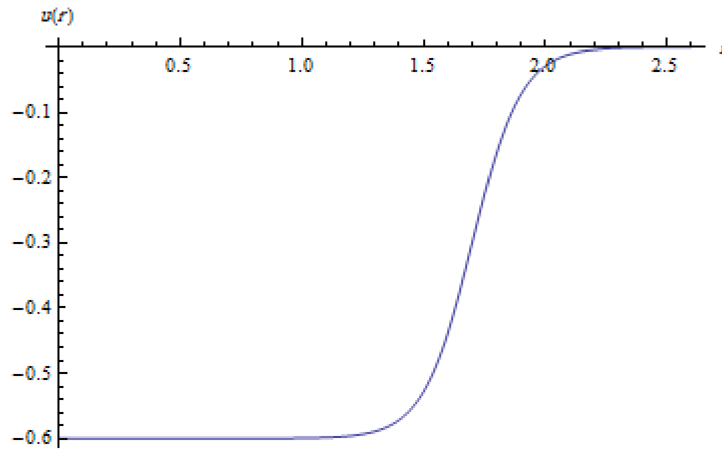
** **

The Saxon - Woods potential

$$\rho = \frac{-\rho 0}{1 + \mathrm{Exp}\left[\frac{r-R}{d}\right]};$$

```
Dρ = D[ρ, r];

Plot[ρ, {r, 0, 2.6}, AxesLabel → {r, u[r]}]
```



Solution of Schrodinger equation in the vicinity of the origin

```
a0 = 1; a1 = 0; a2 = -(k^2 - V0) / (6 (1 - ρ));

v[r_] := Sum[a_n r^(n+1), {n, 0, 2}]

mm = v[r] /. r → 0.0001;

n = D[v[r], r];

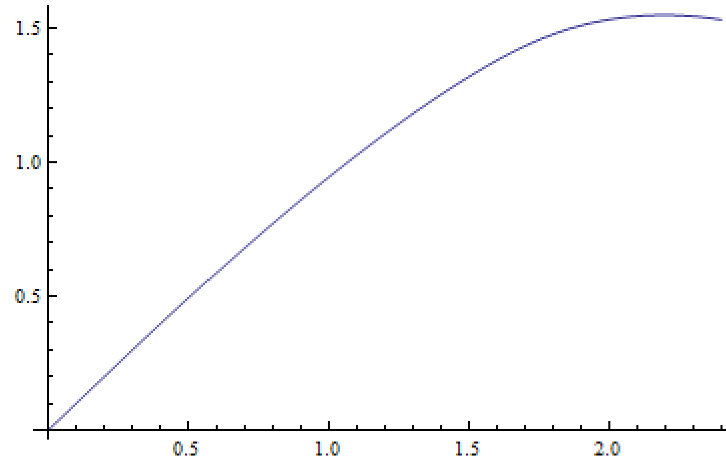
bb = n /. r → 0.0001;
```

** ***

Numerical solution of Schrodinger equation inside the interaction region (r < b)

```
w1 = NDSolve[{ (1 - ρ) u''[r] - (u'[r] - u[r]/r) Dρ + (κ)^2 u[r] == 0,
   u[0.0001] == mm, u'[0.0001] == bb}, u[r], {r, 0.0001, 2.4}]

{{u[r] → InterpolatingFunction[{{0.0001, 2.4}}, <>][r]}}
        PP = Plot[u[r] /. w1, {r, 0.0001, 2.4}]
```




```
w2 = Table[{r, u[r] /. w1[[1]]}, {r, 0.0001, 2.4, 0.0001}];

m = u[r] /. w1 /. r → b;
     w3 = D[u[r] /. w1, r]

     {InterpolatingFunction[{{0.0001, 2.4}}, <>][r]}
    w4 = Table[{r, w3[[1]]}, {r, 0.0001, 2.4, 0.0001}];

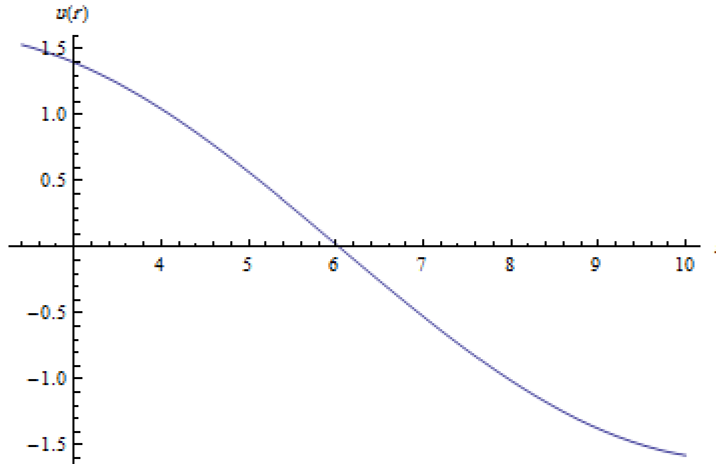
   n = w3 /. r → b;
```

Numerical solution of Schrodinger equation outside the interaction region (r ≥ b)

```
w5 = NDSolve[{ u''[r] + (k)^2 u[r] == 0, u[2.4] == m, u'[2.4] == n},
  u[r], {r, 2.4, 10}]

{{u[r] → InterpolatingFunction[{{2.4, 10.}}, <>][r]}}

PP1 = Plot[u[r] /. w5, {r, 2.4, 10}, AxesLabel → {r, u[r]}]
```




```
Table[{r, u[r] /. w5[[1]]}, {r, 6.0407068`, 6.0407069`, .0000001}]
```

$\{\{6.04071, \{8.96727\times10^{-9}\}\}, \{6.04071, \{-4.6833\times10^{-8}\}\}\}$

```
s = Solve[Sin[k * 6.0407069`] Cos[δ] + Cos[k * 6.0407069`] Sin[δ] == 0, δ]

{{δ → -2.09675}, {δ → 1.04484}}
```

Appendix C
Determining the perturbed phase shifts using the effective range expansion for square well, Saxon Woods potential

```
(*Energy available in the CM of the two interacting nucleons*)En = 5;
(*depth of the local potential*)V0 = 16.9;
(*nucleon mass*)M = 938.0;
(*Planck constant*)hc = 197.3;
(*range of the potential*)b = 2.4;
(*change in the energy*)ΔEn = 0.5;
(*unperturbed phase shift*)δ0 = 1.0448428840635506`;
```

$$k = \sqrt{\frac{M\,En}{(hc)\,^\wedge 2}};$$

$$\kappa = \sqrt{\frac{M\,(En + V0)}{(hc)\,^\wedge 2}};$$

```
(*parametrs of Saxon-Woods potential*)
ρ0 = 0.6;
R = 1.7;
d = 0.1;
```

** ***

The Saxon - Woods potential

$$\rho = \frac{-\rho 0}{1 + \mathrm{Exp}\left[\frac{r-R}{d}\right]};$$

```
Dρ = D[ρ, r];
```

** ***

The asymptotic form of the unperturbed wave function

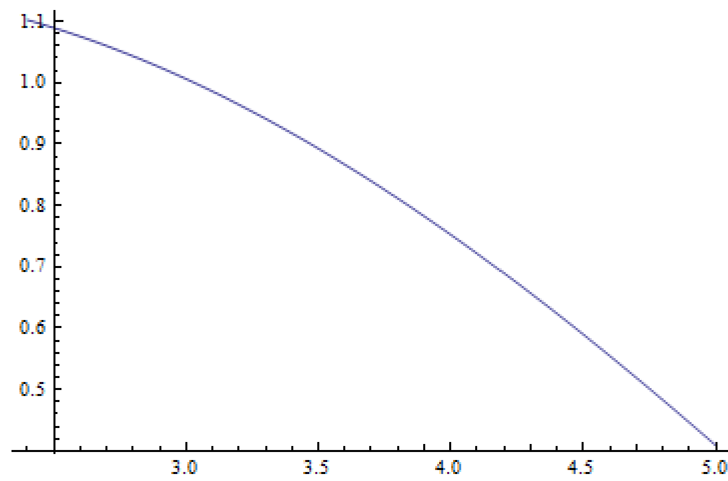
$$y0e = \frac{\mathrm{Sin}[k\,r + \delta 0]}{\mathrm{Sin}[\delta 0]};$$

```
n = y0e /. r → b;

y0e' = D[y0e, r];

y0e'' = D[y0e', r];
```

```
p2 = Plot[y0e, {r, 2.4, 5}]
```



** ** ** ** ** ** ** ** ** ** ** ** ***

Solution of Schrodinger equation in the vicinity of the origin

$$a_0 = 1;\ a_1 = 0;\ a_2 = \frac{-(k\text{^}2 - V0)}{6\ (1 - \rho)};$$

$$v[r_] := \sum_{n=0}^{2} a_n\ r^{n+1}$$

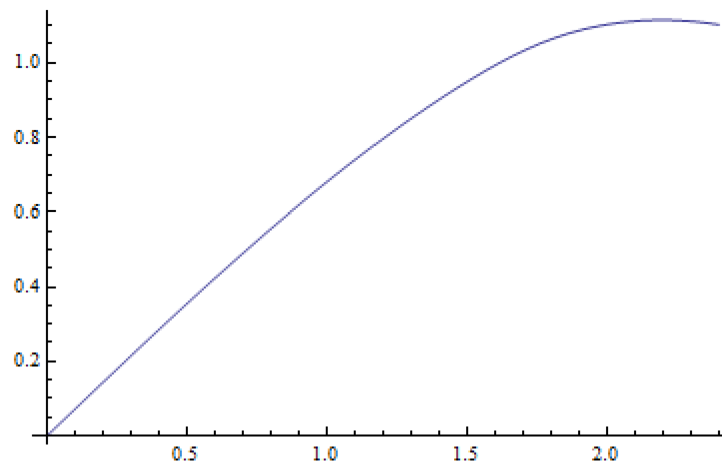
```
mm = v[r] /. r → 0.0001;
```

** ** ** ** ** ** ** ** **

Numerical solution of Schrodinger equation inside the interaction region (r < b)

$$w1 = \text{NDSolve}\left[\left\{ (1 - \rho)\ u0i''[r] - \left(u0i'[r] - \frac{u0i[r]}{r}\right) D\rho + (\kappa)\text{^}2\ u0i[r] == 0,\ u0i[0.0001] == mm,\ u0i[b] == n\right\},\ u0i[r],\ \{r, 0.0001, 2.4\}\right]$$

```
{{u0i[r] → InterpolatingFunction[{{0.0001, 2.4}}, <>][r]}}
yi = u0i[r] /. w1
{InterpolatingFunction[{{0.0001, 2.4}}, <>][r]}
p1 = Plot[yi, {r, 0.0001, 2.4}]
```



```
c = yi /. r → b;

yi' = D[yi, r]

{InterpolatingFunction[{{0.0001, 2.4}}, <>][r]}
                         cc = yi' /. r → 2.4;
yi'' = D[yi', r]

{InterpolatingFunction[{{0.0001, 2.4}}, <>][r]}
```

** ***

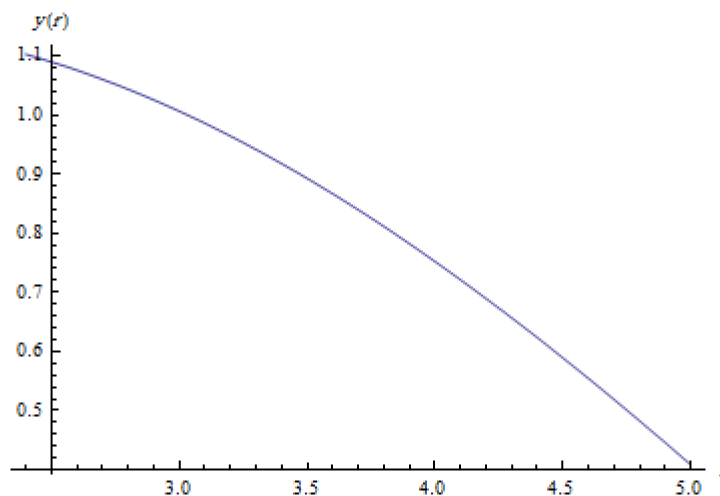
Numerical solution of Schrodinger equation outside the interaction region (r ≥ b)

```
w2 =  NDSolve[{ u0e''[r] + (k)^2 u0e[r] == 0, u0e[2.4] == c, u0e'[2.4] == cc},
  u0e[r], {r, 2.4, 5}]

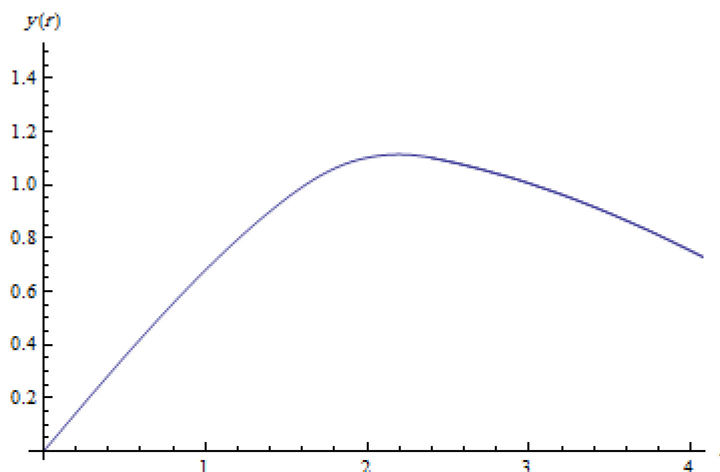
{{u0e[r] → InterpolatingFunction[{{2.4, 5.}}, <>][r]}}

ye = u0e[r] /. w2

{InterpolatingFunction[{{2.4, 5.}}, <>][r]}
p3 = Plot[ye, {r, 2.4, 5}, AxesLabel → {r, y[r]}]
```




```
Show[{p1, p2, p3}, PlotRange → {{0, 4}, {0, 1.5}} , AxesLabel → {r, y[r]}]
```



```
ye' = D[ye, r]

{InterpolatingFunction[{{2.4, 5.}}, <>][r]}

ye'' = D[ye', r]

{InterpolatingFunction[{{2.4, 5.}}, <>][r]}

** ** ** ** ** ** ** ** ** ** ** ** ** ** ** ** ** ** ** ** ** ** ** ***
```

To calculate the effective range parameter α_1

```
alpha1 = (NIntegrate[y0e^2, {r, 0, 2.4}] - NIntegrate[yi^2, {r, 0, 2.4}])
  (M / (hc)^2)

{0.0351335}
```

An analyitical expression of the first - order wave function in the asymptotic region

$$\delta 0' = 2\,\mathrm{Sin}[\delta 0]^2 \int_0^b (y0e)^2\, dr + \frac{\mathrm{Sin}[2\,k\,b + 2\,\delta 0]}{2\,k} - b;$$

$$v1e = \left(\frac{r\,\mathrm{Cos}[k\,r + \delta 0]}{2\,k\,\mathrm{Sin}[\delta 0]} - \frac{\delta 0'\,\mathrm{Sin}[k\,r]}{2\,k\,\mathrm{Sin}[\delta 0]^2}\right)\left(\frac{M}{(hc)^2}\right);$$

```
a = v1e /. r → b;

D[v1e, r] /. r → b;
```

$$k1 = \sqrt{\frac{M\,(En + \Delta En)}{(hc)^2}};$$

$$\kappa 1 = \sqrt{\frac{M\,(En + V0 + \Delta En)}{(hc)^2}};$$

```
** ** ** ** ** ** ** ** ** ** ** ** ** ** ** ** ** ** ***
```

The first - order wave function correction in the internal region (r < b)

```
s1
 = NDSolve[
  {(1 - ρ) (yi u1i''[r] - yi'' u1i[r]) + (u1i[r] yi' - yi u1i'[r]) Dρ +
      (M / (hc)^2) yi^2 == 0, u1i[b] == a, u1i[0.0001] == mm}, u1i[r],
  {r, 0.0001, 2.4}]

{{u1i[r] → InterpolatingFunction[{{0.0001, 2.4}}, <>][r]}}
```

```
y1i = u1i[r] /. s1
{InterpolatingFunction[{{0.0001, 2.4}}, <>][r]}

t = y1i /. r → b;

y1i' = D[y1i, r]
{InterpolatingFunction[{{0.0001, 2.4}}, <>][r]}

tt = y1i' /. r → b;
```

** ***

To calculate the shape parameter α_2

```
alpha2 =
 (2 NIntegrate[v1e y0e, {r, 0, 2.4}] - 2 NIntegrate[yi y1i, {r, 0, 2.4}])
  (M / (hc)^2)
{0.000156501}
```

** **

To calculate the perturbed phase shift δ^P using the effective range expansion (kcotδ)

```
alpha0 = k Cot[δ0];

kcot[δ] = alpha0 + (ΔEn) alpha1 + (ΔEn)^2 (alpha2 / 2)
{0.219078}

a = kcot[δ] / k1
{0.601787}

ArcCot[a]
{1.02906}
```

The perturbed phase shift δ^P = 1.029063569013096`

نظرية الترجاف في تشتت نظام بروتون – نيوترون: الترجاف في الطاقة.

إعداد
محمد إبراهيم محمد أبو السيد

المشرف
الدكتور محمود إسماعيل الجاغوب

ملخـــــص
تم في هذه الرسالة دراسة معادلة شرودنغر غير المعتمدة على الزمن والتي تحتوي على حد كسلنجر المعتمد على السرعة. وقد تم اشتقاق صيغ للتغيرات في اقتران الموجة عندما يحدث اضطراب في طاقة الموجة الساقطة. كما تم أيضا دراسة التغيرات في الازاحة الطورية عند تغير طاقة الموجة الساقطة عن طريق نظرية المدى الفعال.
وللتحقق من صحة الصيغ الرياضية المشتقة في هذه الرسالة ودقة نتائجها فقد تم حل معادلة شرودنغر في ثلاث حالات: الحالة الأولى تم افتراض كل من الجهد الموضعي وحد كسلنجر على شكل جهد بئري محدد الأبعاد بنصف قطر. أما الحالة الثانية فقد تم افتراض أن الجهد الموضعي كما في الحالة الأولى أما جهد كسلنجر فقد تم افتراضه على أنه جهد ساكسون وودز. وفي الحالة الثالثة تم افتراض أن كل من الجهد الموضعي وحد كسلنجر على شكل جهد ساكسون وودز. في الحالة الأولى تم حل معادلة شرودنغر بدون تقريب وتم حلها باستخدام نظرة الترجاف، أما في الحالة الثانية والثالثة فقد تم حل معادلة شرودنغر بشكل عددي ثم حلها باستخدام نظرية الترجاف.
وفي جميع الحالات فإن التغيرات في الازاحة الطورية التي تم حسابها بالطريقة العددية ونظرية الترجاف كانت متقاربة جدا.